

ON THE ZEROS OF CERTAIN MODULAR FUNCTIONS FOR THE NORMALIZERS OF CONGRUENCE SUBGROUPS OF LOW LEVELS I

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Abstract. We research the location of the zeros of the Eisenstein series and the modular functions from the Hecke type Faber polynomials associated with the normalizers of congruence subgroups which are of genus zero and of level at most twelve.

In Part I, we will consider the general theory of modular functions for the normalizers.

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INTRODUCTION

The motive of this research is to decide the location of the zeros of modular functions. The Eisenstein series and the Hecke type Faber polynomials are the most interesting and important modular forms.

F. K. C. Rankin and H. P. F. Swinnerton-Dyer considered the problem of locating the zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain \mathbb{F} (See [RSD]). They proved that all of the zeros of $E_k(z)$ in \mathbb{F} lie on the unit circle. They also stated towards the end of their study that “This method can equally well be applied to Eisenstein series associated with subgroups of the modular group.” However, it seems unclear how widely this claim holds.

Subsequently, T. Mieazaki, H. Nozaki, and the present author considered the same problem for the Fricke group $\Gamma_0^*(p)$ (see [Kr], [Q]), and proved that all of the zeros of the Eisenstein series $E_{k,p}^*(z)$ in a certain fundamental domain lie on a circle whose radius is equal to $1/\sqrt{p}$, $p = 2, 3$ (see [MNS]). Furthermore, we also proved that almost all the zeros of the Eisenstein series in a certain fundamental domain lie on circles whose radius are equal to $1/\sqrt{p}$ or $1/(2\sqrt{p})$, $p = 5, 7$ (see [SJ2]).

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and let h be the width of Γ , then we define

$$(1) \quad \mathbb{F}_{0,\Gamma} := \left\{ z \in \mathbb{H}; -h/2 < \operatorname{Re}(z) < h/2, |cz + d| > 1 \text{ for } \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ s.t. } c \neq 0 \right\}.$$

We have a fundamental domain \mathbb{F}_Γ such that $\mathbb{F}_{0,\Gamma} \subset \mathbb{F}_\Gamma \subset \overline{\mathbb{F}_{0,\Gamma}}$. Let \mathbb{F}_Γ be such a fundamental domain.

For the modular group $\mathrm{SL}_2(\mathbb{Z})$ and the Fricke groups $\Gamma_0^*(p)$ ($p = 2, 3$), all the zeros of the Eisenstein series for the cusp ∞ lie on the arcs on the boundary of their certain fundamental domains.

H. Hahn considered that the location of the zeros of the Eisenstein series for the cusp ∞ for every genus zero Fuchsian group Γ of the first kind with ∞ as a cusp which satisfies that its hauptmodul J_Γ takes real value on $\partial\mathbb{F}_\Gamma$, and proved that almost all the zeros of the Eisenstein series for the cusp ∞ for Γ lie on $\partial\mathbb{F}_\Gamma$ under some more assumption (see [H]).

Also, T. Asai, M. Kaneko, and H. Ninomiya considered the problem of locating the zeros of modular functions $F_m(z)$ for $\mathrm{SL}_2(\mathbb{Z})$ which correspond to the Hecke type Faber polynomial P_m , that is, $F_m(z) = P_m(J(z))$ (See [AKN]). They proved that all of the zeros of $F_m(z)$ in \mathbb{F} lie on the unit circle for each $m \geq 1$. After that, E. Bannai, K. Kojima, and T. Mieazaki considered the same problem for the normalizers of congruence subgroups which correspond the conjugacy classes of the Monster group (See [BKM]). They observed the location of the zeros by numerical calculation, then almost all of the zeros of the modular functions from Hecke type Faber polynomial lie on the lower arcs when the group satisfy the same assumption of the theorem of H. Hahn. In particular, T. Mieazaki proved that all of the zeros of the modular functions from the Hecke type Faber polynomials for the Fricke group $\Gamma_0^*(2)$ lie on the lower arcs of its fundamental domain in their paper.

Now, we have the following conjectures:

Conjecture 1. *Let Γ be a genus zero Fuchsian group of the first kind with ∞ as a cusp. If the hauptmodul J_Γ takes real value on $\partial\mathbb{F}_\Gamma$, all of the zeros of the Eisenstein series for the cusp ∞ for Γ in \mathbb{F}_Γ lie on the*

arcs

$$\partial\mathbb{F}_\Gamma \setminus \{z \in \mathbb{H}; \operatorname{Re}(z) = \pm h/2\}.$$

Conjecture 2. *Let Γ be a genus zero Fuchsian group of the first kind with ∞ as a cusp. If the hauptmodul J_Γ takes real value on $\partial\mathbb{F}_\Gamma$, all but at most $c_h(\Gamma)$ of the zeros of modular function from the Hecke type Faber polynomial of degree m for Γ in \mathbb{F}_Γ lie on the arcs*

$$\partial\mathbb{F}_\Gamma \setminus \{z \in \mathbb{H}; \operatorname{Re}(z) = \pm h/2\}$$

for all but finite number of m and for the constant number $c_h(\Gamma)$ which does not depend on m .

In this paper, we will observe the location of the zeros of the Eisenstein series and the modular functions from Hecke type Faber polynomials for the normalizers of congruence subgroups, as a first step of a challenge for the above conjectures.

The normalizers of congruence subgroups of level at most 12 which satisfies the assumption of above conjectures are

$$\begin{aligned} & \mathrm{SL}_2(\mathbb{Z}), \Gamma_0^*(2), \Gamma_0(2), \Gamma_0^*(3), \Gamma_0(3), \Gamma_0^*(4), \Gamma_0(4), \Gamma_0^*(5), \Gamma_0(6)+, \Gamma_0^*(6), \Gamma_0(6)+3, \Gamma_0(6), \\ & \Gamma_0^*(7), \Gamma_0^*(8), \Gamma_0(8), \Gamma_0^*(9), \Gamma_0(10)+, \Gamma_0^*(10), \Gamma_0(10)+5, \Gamma_0(12)+, \Gamma_0^*(12), \\ & \Gamma_0(12)+4, \text{ and } \Gamma_0(12). \end{aligned}$$

For the Conjecture 1, $\mathrm{SL}_2(\mathbb{Z})$, $\Gamma_0^*(2)$, and $\Gamma_0^*(3)$ verify Conjecture 1. For the other cases, we can prove by numerical calculation for the Eisenstein series of weight $k \leq 500$.

For the Conjecture 2, $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0^*(2)$ verify Conjecture 2 for every degree m , where we have $c_h(\Gamma) = 0$ for each case. Furthermore, for $\Gamma_0(2)$, $\Gamma_0^*(3)$, $\Gamma_0(3)$, $\Gamma_0^*(4)$, $\Gamma_0(4)$, $\Gamma_0(6)+$, $\Gamma_0(6)+3$, $\Gamma_0(6)$, $\Gamma_0(8)$, $\Gamma_0^*(9)$, $\Gamma_0(10)+$, $\Gamma_0(10)+5$, $\Gamma_0(12)+$, $\Gamma_0(12)+4$, and $\Gamma_0(12)$, we can prove all of the zeros of the modular function from the Hecke type Faber polynomial of every degree $m \leq 200$ in each fundamental domain lie on the lower arcs by numerical calculation.

On the other hand, for $\Gamma_0^*(5)$ and $\Gamma_0^*(7)$, we can prove by numerical calculation for the modular function from the Hecke type Faber polynomial of every degree $m = 1$ and $3 \leq m \leq 200$, where we have $c_h(\Gamma) = 0$ for each case. When $m = 2$, there is just one zero which is on the boundary of its fundamental domain but not on the lower arcs for the each group.

For $\Gamma_0^*(6)$ and $\Gamma_0^*(8)$, we can prove by numerical calculation for the modular function from the Hecke type Faber polynomial of every degree $m \leq 200$ which satisfy $m \not\equiv 0 \pmod{2}$ and $m \not\equiv 2 \pmod{4}$, respectively. For the remaining degrees, there is just one zero which is on the boundary of its fundamental domain but not on the lower arcs for the each group, that is, $c_h(\Gamma) = 1$.

Finally, for $\Gamma_0^*(10)$ and $\Gamma_0^*(12)$, we have just two zeros which are not on the boundary of each fundamental domain for degrees $m = 7, 9, 11$ and $m = 3, 6, 12, 13, 15$, respectively. Furthermore, there is just one zero which is on the boundary of its fundamental domain but not on the lower arcs for the case $m \equiv 0 \pmod{2}$ and $m \equiv 2, 4 \pmod{6}$, respectively. For the other cases, we can prove that all of the zeros are on the lower arcs of each fundamental domain by numerical calculation.

Γ	Eisenstein series ($k \leq 500$)	Hecke type Faber polynomial ($m \leq 200$)
$\mathrm{SL}_2(\mathbb{Z}), \Gamma_0^*(2), \Gamma_0(2), \Gamma_0^*(3), \Gamma_0(3),$ $\Gamma_0^*(4), \Gamma_0(4), \Gamma_0(6)+, \Gamma_0(6)+3,$ $\Gamma_0(6), \Gamma_0(8), \Gamma_0^*(9), \Gamma_0(10)+, \Gamma_0(10)+5,$ $\Gamma_0(12)+, \Gamma_0(12)+4, \Gamma_0(12).$		○
$\Gamma_0^*(5), \Gamma_0^*(7)$	○	$m = 2, \quad \langle 1 \rangle$
$\Gamma_0^*(6)$		$m : \text{even}, \quad \langle 1 \rangle$
$\Gamma_0^*(8)$		$m \equiv 0 \pmod{4}, \quad \langle 1 \rangle$
$\Gamma_0^*(10)$		$m = 7, 9, 11, [2], \quad m : \text{even}, \langle 1 \rangle$
$\Gamma_0^*(12)$		$m = 3, 6, 12, 13, 15, [2] \quad m \equiv 2, 4 \pmod{6}, \langle 1 \rangle$

‘○’: all of the zeros lie on lower arcs.

$\langle \cdot \rangle$: the number of zeros which are on $\partial\mathcal{F}$ but not on lower arcs.

$[\cdot]$: the number of zeros which are not on $\partial\mathcal{F}$.

TABLE 1. Result by numerical calculation

If the hauptmodul J_Γ does not take real value on $\partial\mathbb{F}_\Gamma$ (cf. Figure 1), it seems to be not similar. Such cases are followings;

$$\Gamma_0(5), \Gamma_0(6) + 2, \Gamma_0(7), \Gamma_0(9), \Gamma_0(10) + 2, \Gamma_0(10), \Gamma_0^*(11), \text{ and } \Gamma_0(12) + 3.$$

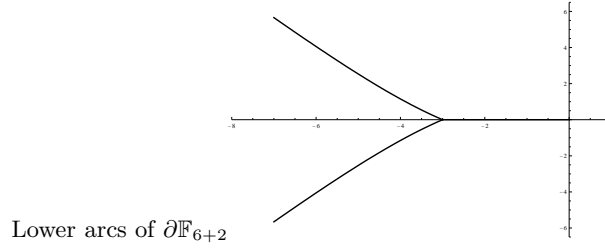


FIGURE 1. Image by J_{6+2} ($\Gamma_0(6) + 2$)

For $\Gamma_0(5)$, $\Gamma_0(6) + 2$, $\Gamma_0(7)$, $\Gamma_0(10) + 2$, $\Gamma_0(10)$, and $\Gamma_0^*(11)$, we can observe that the zeros of the Eisenstein series for cusp ∞ do not lie on the lower arcs of their fundamental domains by numerical calculation. However, when the weight of Eisenstein series increases, then the location of the zeros seems to approach to lower arcs. (See Figure 2)

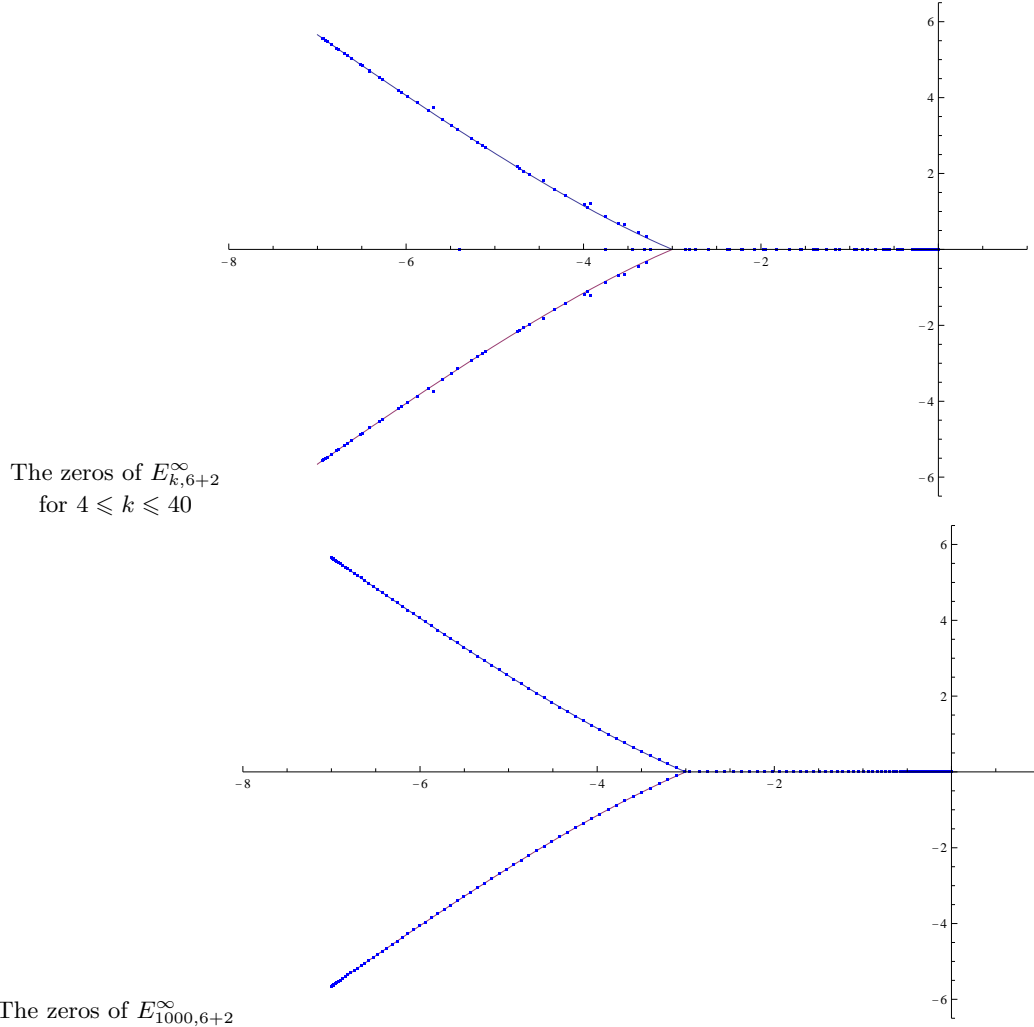


FIGURE 2. Image by J_{6+2} ($\Gamma_0(6) + 2$)

Also, for the zeros of the modular functions from the Hecke type Faber polynomials, we can observe that there are some zeros which do not lie on the lower arcs of their fundamental domains by numerical calculation. Furthermore, when the degree m increases, then the location of the zeros seems to approach to lower arcs. (See Figure 3)

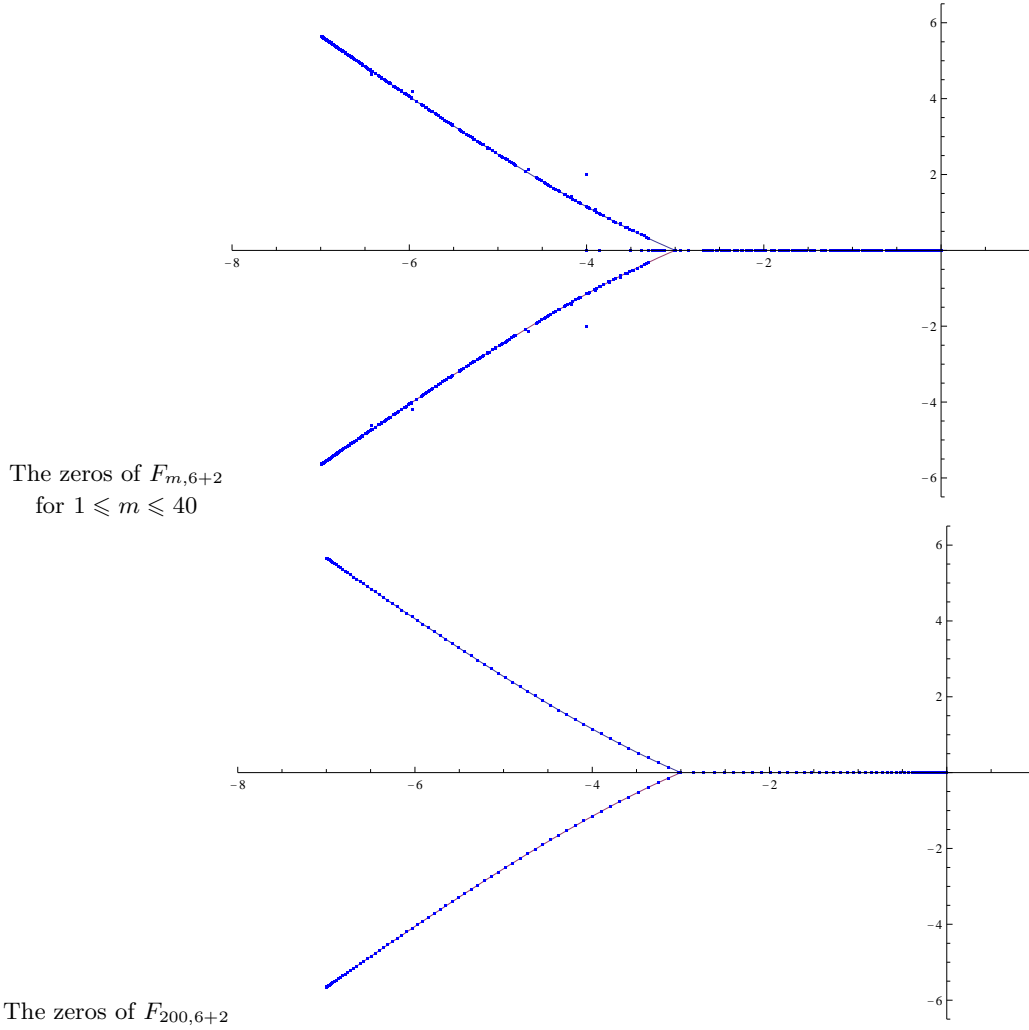


FIGURE 3. Image by $J_{6+2}(\Gamma_0(6) + 2)$

On the other hand, $\Gamma_0(9)$ and $\Gamma_0(12) + 3$ seem to show the special cases. We can prove that all of the zeros of the Eisenstein series of weight $k \leq 500$ lie on the lower arcs of their fundamental domains by numerical calculation. Also, we can prove that all of the zeros of the modular function from the Hecke type Faber polynomial of degree $m \leq 200$ lie on the lower arcs by numerical calculation. On the other hand, they do not satisfy the assumption of Conjecture 1 and 2. However, the image of lower arcs by its hauptmodul draw a interesting figure. (Figure 4)

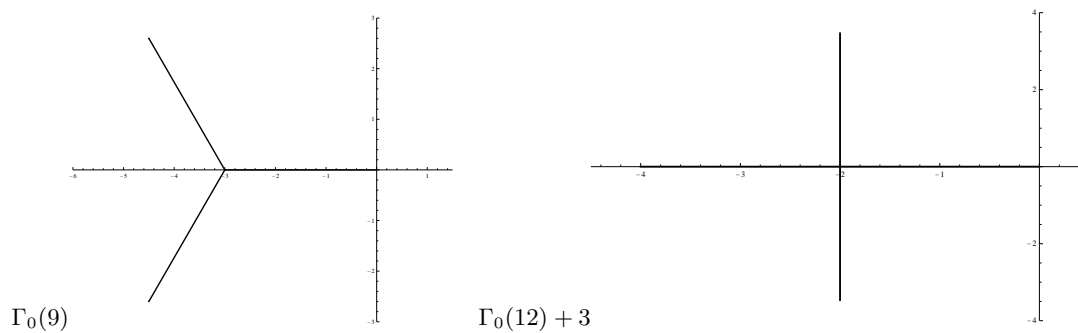


FIGURE 4. Image of the lower arcs of the fundamental domains by hauptmoduls

We refer to [MNS], [SJ1], and [SJ2] for some groups. However, note that definitions in this paper are sometimes different from that in it.

In ‘Part I’, we will consider the general theory of modular functions for the normalizers of the congruence subgroups $\Gamma_0(N)$ of level $N \leq 12$. And in ‘Part II’, we will observe the location of the zeros of the Eisenstein series and the modular functions from Hecke type Faber polynomials for the normalizers in Part I by numerical calculation.

0. GENERAL THEORY

Let Γ be a Fuchsian group of the first kind with ∞ as a cusp.

0.1. The modular group and some groups.

0.1.1. *The modular group.* (see [Se, VII.1], [SH, I-I], and [Ko, III.1])

We have the *special linear group* defined by following:

$$(2) \quad \mathrm{SL}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \forall a, b, c, d \in \mathbb{R} \text{ s.t. } ad - bc = 1 \right\}.$$

Write $\mathbb{H} := \{z \in \mathbb{C} ; \operatorname{Im}(z) > 0\}$, which is complex upper half-plane. We consider a action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H} \cup (\{\infty\} \cup \mathbb{R})$ in the following way:

For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and every $z \in \mathbb{H}$, we put

$$(3) \quad \gamma z := \frac{az + b}{cz + d}.$$

Note that $-\gamma z = \gamma z$ and $\operatorname{Im}(\gamma z) = \operatorname{Im}(z)/|cz + d|^2$ for every $z \in \mathbb{H}$.

We also define

$$(4) \quad \mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) ; \forall a, b, c, d \in \mathbb{Z} \right\},$$

which is called the (*full*) *modular group*.

0.1.2. *Congruence subgroup.* (see [Ko, III.1])

For a positive integer N , we have

$$(5) \quad \Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) ; a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N} \right\}.$$

This group is a subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, which is called the *principal congruence subgroup of level N* .

Also, if Γ' is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma' \supset \Gamma(N)$, then Γ' is called a *congruence subgroup of level N* . Here are some examples:

$$(6) \quad \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) ; c \equiv 0 \pmod{N} \right\},$$

$$(7) \quad \Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) ; a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\}.$$

0.1.3. *Fricke group.* (see [Kr], [Q])

For a positive integer N , we consider the *Fricke group* $\Gamma_0^*(N)$. We define the following;

$$(8) \quad \Gamma_0^*(N) := \Gamma_0(N) \cup \Gamma_0(N) W_N, \quad W_N := \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

This group is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and commensurable with $\mathrm{SL}_2(\mathbb{Z})$.

0.1.4. *The normalizer of $\Gamma_0(N)$.* (see [CN])

We introduce the normalizer of $\Gamma_0(N)$. About notations, there is something different from [CN].

We fix positive integers n and h such that $h|n$ and $h|24$. Then, we define

$$(9) \quad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}_e := \frac{1}{\sqrt{e}} \begin{pmatrix} ae & b/h \\ cn & de \end{pmatrix}$$

for the integers $e > 0$ and a, b, c, d such that $e|(n/h)$ and $ade - bc(n/h)/e = 1$. We denote $I_{n|h} := \{e \in \mathbb{N} ; e|(n/h)\}$, where $n|m$ means that $n|m$ and $(n, m/n) = 1$.

Then, we define

$$(10) \quad \Gamma_0(n|h) := \left\{ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}_1 ; a, b, c, d \in \mathbb{Z} \text{ s.t. } ad - bc(n/h) = 1 \right\}.$$

$$(11) \quad \Gamma_0(n|h) + e_1, e_2, \dots, e_m := \left\{ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}_e ; \begin{array}{l} a, b, c, d \in \mathbb{Z} \text{ s.t. } ade - bc(n/h)/e = 1 \\ e \in \{1, e_1, \dots, e_m\} \end{array} \right\},$$

where $\{1, e_1, \dots, e_m\} \subset I_{n|h}$.

$$(12) \quad \Gamma_0(n|h)_+ := \left\{ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}_e ; \begin{array}{l} a, b, c, d \in \mathbb{Z} \text{ s.t. } ade - bc(n/h)/e = 1 \\ e \in I_{n|h} \end{array} \right\}.$$

In addition, we have $\Gamma_0(n) = \Gamma_0(n|1)$ and denote $\Gamma_0(n|h)- := \Gamma_0(n|h)$.

For example, for a prime number p , we consider the case $h = 1$ and $n = p$, then we have $I_{p|1} = \{1, p\}$ and

$$\begin{aligned} \left\{ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right\}_1 &= \left(\begin{smallmatrix} a & b \\ cp & d \end{smallmatrix} \right) \in \Gamma_0(p), \\ \left\{ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right\}_p &= \left(\begin{smallmatrix} a\sqrt{p} & b/\sqrt{p} \\ c\sqrt{p} & d/\sqrt{p} \end{smallmatrix} \right) = \left(\begin{smallmatrix} -b & a \\ -dp & c \end{smallmatrix} \right) \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} \in \Gamma_0(p)W_p. \end{aligned}$$

Thus, we have $\Gamma_0(p)+ = \Gamma_0(p) + p = \Gamma_0^*(p)$ and $\Gamma_0(p)- = \Gamma_0(p)$. That is, Fricke group $\Gamma_0^*(p)$ is a normalizers of $\Gamma_0(p)$.

0.1.5. *Preliminaries.* Write $T_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} (\in \mathrm{SL}_2(\mathbb{R}))$ and $P := \{\pm T_x; x \in \mathbb{R}\}$. In this paper, we may assume

$$(13) \quad \Gamma \cap P \setminus \{\pm I\} \neq \emptyset$$

and $\Gamma \cap P$ is discrete. Then, we call $h := \min\{x > 0; T_x \in \Gamma\}$ the *width* of Γ .

For a cusp κ of Γ , we define *the stabilizer of the cusp* κ :

$$\Gamma_\kappa := \{\gamma \in \Gamma; \gamma\kappa = \kappa\}.$$

In particular, we have $\Gamma_\infty = \Gamma \cap P = \{T_{nh}; n \in \mathbb{Z}\}$. Furthermore, there exist some $\gamma_\kappa \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma_\kappa \infty = \kappa$ and

$$\Gamma_\kappa = \gamma_\kappa \Gamma_\infty \gamma_\kappa^{-1}.$$

We call γ_κ the *cuspid leader* of the cusp κ . Note that, Γ_κ and γ_κ are depend on the group Γ .

0.2. **Fundamental domain.** (see [Se, VII.1], [SH, I-I.4])

In this section, we consider a fundamental domain in \mathbb{H} under the action of Γ (equation (3)).

Definition 0.1. \mathbb{F}_Γ is a *fundamental domain* of Γ if and only if it satisfies following conditions:

(FD1) For every $z \in \mathbb{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathbb{F}_\Gamma$.

(FD2) For every two distinct points $z_1, z_2 \in \mathbb{F}_\Gamma$, there does not exist $\gamma \in \Gamma$ such that $\gamma z_1 = z_2$.

We define

$$\mathbb{F}_{0,\Gamma} := \{z \in \mathbb{H}; -h/2 < \mathrm{Re}(z) < h/2, |cz + d| > 1 \text{ for } \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus P\}$$

(see [SH, I-I.4 & 5, Theorem 1.7 and 1.15]). Now, we have the following fact:

Proposition 0.1.

(i) $\overline{\mathbb{F}_{0,\Gamma}}$ satisfies the condition (FD1).

(ii) $\mathbb{F}_{0,\Gamma}$ satisfies the condition (FD2).

Thus, we have a fundamental domain \mathbb{F}_Γ such that $\mathbb{F}_{0,\Gamma} \subset \mathbb{F}_\Gamma \subset \overline{\mathbb{F}_{0,\Gamma}}$. Note that, let $z, z' \in \partial \mathbb{F}_{0,\Gamma}$. If $\gamma z = z'$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus P$, then $|cz + d| = 1$. Also, we have $z = (e^{i\theta} - d)/c$ and $z' = (e^{i(\pi-\theta)} + a)/c$.

We define $\Gamma^0 := \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma; |cz + d| = 1 \text{ for } \exists z \in \mathbb{F}_{0,\Gamma}\}$, then we have the following fact:

Corollary 0.1.1. *If $\Gamma \ni -I$, then $\Gamma^0 \cup \{T_h, -I\}$ generates Γ . On the other hand, if $\Gamma \not\ni -I$, $\Gamma^0 \cup \{T_h\}$ generates Γ .*

0.3. Modular forms.

0.3.1. *Preliminaries.* (see [Ko, III.2, 3] and [Se, VII.1])

Let f be a function on \mathbb{H} . For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, we denote

$$(14) \quad f|_k \gamma (z) := (cz + d)^{-k} f(\gamma z).$$

Then, the relation

$$(15) \quad f|_k \gamma (z) = f(z) \quad \text{for every } z \in \mathbb{H} \text{ and every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is called the *transformation rule* for Γ .

Incidentally, since $\mathrm{SL}_2(\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$, transformation rule for $\mathrm{SL}_2(\mathbb{Z})$ is equivalent to the following two equations:

$$(16) \quad f(z+1) = f(z),$$

$$(17) \quad f\left(-\frac{1}{z}\right) = z^k f(z).$$

We have the Fourier expansion:

$$(18) \quad f(z) = \sum_{n \in \mathbb{Z}} a_n q_h^n, \quad \text{where } q_h = e^{2\pi iz/h}.$$

Similarly, we have the following Fourier expansion for every cusp κ of Γ with the cusp leader γ_κ :

$$(19) \quad f|_k \gamma_\kappa(z) = \sum_{n \in \mathbb{Z}} a_{\kappa,n} q_h^n, \quad \text{where } q_h = e^{2\pi iz/h}.$$

When $h = 1$, we denote $q = q_1$. We say f is *meromorphic at the cusp* κ if $a_{\kappa,n}$ is zero for n small enough. Also, we call f *holomorphic at the cusp* κ if $a_{\kappa,n}$ is zero for every negative integer n .

Definition 0.2. Let f be a meromorphic function on \mathbb{H} . f is called a *modular function* for Γ if f is meromorphic at every cusp and satisfies transformation rule for Γ .

For a meromorphic function f , we assume $f(\kappa) = 0$ if and only if $a_{\kappa,n} = 0$ for every integer $n \leq 0$.

Definition 0.3. Let f be a modular function for Γ which is holomorphic on \mathbb{H} , then f is called *weakly modular form* for Γ . In addition, if f is holomorphic at every cusp of Γ , then f is called *modular form* for Γ . Furthermore, if f is equal to 0 at every cusp of Γ , we call f *cusp form* for Γ .

For a function f , let $v_p(f)$ be the order of f at $p \in \mathbb{H}$. In addition, we also define the order of f at a cusp κ :

$$v_\kappa(f) := \min\{n \in \mathbb{Z}; a_{\kappa,n} \neq 0\}.$$

Furthermore, we have following facts:

Proposition 0.2. Let f be a modular form for Γ such that every coefficient of Fourier expansion is real. Then we have $f(-\bar{z}) = \overline{f(z)}$ and $v_\rho(f) = v_{-\bar{\rho}}(f)$ at $\rho \in \mathbb{H}$.

Let f be a modular function for Γ of weight k . If we have $\gamma\Gamma\gamma^{-1} \subset \Gamma$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, then we have

$$f|_k \gamma (\gamma' \gamma^{-1} z) = ((c'd - cd')z + (ad' - bc'))^k f(z)$$

for every $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma$. Then, we have

$$f|_k \gamma (\gamma' z) = (c'z + d')^k (cz + d)^{-k} f(\gamma z) = (c'z + d')^k f|_k \gamma (z).$$

Thus, $f|_k \gamma$ is also a modular function for Γ of weight k .

0.3.2. Hauptmodul.

Let Γ be of genus 0. Then, we consider the weakly modular form for Γ which is holomorphic at every cusp but ∞ and has the following form of Fourier expansion:

$$(20) \quad j_\Gamma(z) := \frac{1}{q_h} + \sum_{n=0}^{\infty} a_n q_h^n.$$

It is determined uniquely up to the constant term. We call j_Γ (*canonical*) *hauptmodul* of Γ . Note that it is a isomorphism from a fundamental domain \mathcal{F}_Γ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Similarly, we can define *hauptmodul for the cusp* κ j_Γ^κ which has the following form of Fourier expansion for the cusp κ :

$$(21) \quad j_\Gamma^\kappa|_0 \gamma_\kappa(z) = \frac{1}{q_h} + \sum_{n=0}^{\infty} a_{n,\kappa} q_h^n.$$

Note that we have $j_\Gamma^\kappa(z) = j_\Gamma(\gamma_\kappa^{-1}z)$ if $\gamma_\kappa^{-1}\Gamma\gamma_\kappa \subset \Gamma$. In this paper, we consider only *hauptmodul* for the cusp ∞ .

0.3.3. *Eisenstein series.* (see [SG])

Definition 0.4. For $z \in \mathbb{H}$,

$$(22) \quad E_{k,\Gamma}^\kappa(z) := e \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} j(\gamma_\kappa^{-1}\gamma, z)^{-k} \quad (e : \text{fixed number})$$

is the *Eisenstein series* associated with Γ for a cusp κ , where $j(\gamma, z) := cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. e is often selected so that the constant term of $E_{k,\Gamma}^\kappa$ is 1.

For example, let $\Gamma = \text{SL}_2(\mathbb{Z})$, then we have only ∞ as a cusp of $\text{SL}_2(\mathbb{Z})$. Now, for an even integer $k \geq 4$, we have

$$(23) \quad E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k}$$

as the Eisenstein series associated with $\text{SL}_2(\mathbb{Z})$.

For $k = 2$, we can define $E_2(z)$ as $E_k(z)$ for $k \geq 4$.

$$(24) \quad E_2(z) := \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-2} = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \dots$$

$E_2(z)$ is not modular form for $\text{SL}_2(\mathbb{Z})$. $E_2(z)$ is holomorphic on \mathbb{H} and at ∞ , and it satisfies transformation rule (16). On the other hand, it does not satisfy (17), instead, we have

$$(25) \quad E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) + \frac{12}{2\pi i} z.$$

If $\gamma_\kappa^{-1}\Gamma\gamma_\kappa = \Gamma$, then we have some relation between the Eisenstein series $E_{k,\Gamma}^\infty$ and $E_{k,\Gamma}^\kappa$. By the residue class $\Gamma_\kappa \setminus \Gamma$, we have

$$\Gamma = \bigcup_{\gamma \in \Gamma_\kappa \setminus \Gamma} \Gamma_\kappa \cdot \gamma = \bigcup_{\gamma \in \Gamma_\kappa \setminus \Gamma} (\gamma_\kappa \Gamma_\infty \gamma_\kappa^{-1}) \cdot \gamma, \quad \Gamma = \gamma_\kappa^{-1}\Gamma\gamma_\kappa = \bigcup_{\gamma \in \Gamma_\kappa \setminus \Gamma} \Gamma_\infty \cdot (\gamma_\kappa^{-1}\gamma\gamma_\kappa).$$

Then, it gives us a residue class $\Gamma_\infty \setminus \Gamma$. Second, we have $j(\gamma_\kappa^{-1}\gamma, \gamma_\kappa z) = j(\gamma_\kappa, z)^{-1} j(\gamma_\kappa^{-1}\gamma\gamma_\kappa, z)$. Thus, we have

$$\begin{aligned} E_{k,\Gamma}^\kappa(\gamma_\kappa z) &= e \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} j(\gamma_\kappa^{-1}\gamma, \gamma_\kappa z)^{-k} = j(\gamma_\kappa, z)^k \cdot e \sum_{\gamma \in \Gamma_\kappa \setminus \Gamma} j(\gamma_\kappa^{-1}\gamma\gamma_\kappa, z)^{-k} \\ &= e' j(\gamma_\kappa, z)^k \cdot e'' \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, z)^{-k} = e' j(\gamma_\kappa, z)^k E_{k,\Gamma}^\infty(z). \end{aligned}$$

0.3.4. *Eta function.* (see [Ko, III.2])

We put

$$(26) \quad \Delta(z) := \frac{1}{1728} \left((E_4(z))^3 - (E_6(z))^2 \right).$$

Δ is the cusp form for $\text{SL}_2(\mathbb{Z})$ of weight 12 which satisfies $v_\infty(\Delta) = 1$. Now, we have

Theorem 0.3 (Jacobi's product formula).

$$(27) \quad \Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad \text{where } q = e^{2\pi i z}.$$

Also, we have

$$(28) \quad \eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

which is called the *Dedekind η -function*. Then we have

$$(29) \quad \eta(z+1) = e^{\frac{2\pi i}{24}} \eta(z) \quad \text{and} \quad \eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}} \eta(z) \quad (\text{see [Ko]}),$$

where $\sqrt{\cdot}$ denote a square root which has nonnegative real part. Furthermore, we have

$$(30) \quad \eta\left(\frac{az+b}{cz+d}\right) = \epsilon \sqrt{\frac{cz+d}{i}} \eta(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

where ϵ is one of the 24th-roots of 1 which depends on a, b, c , and d .

0.3.5. Hecke type Faber polynomial. (see [ACMS])

Let Γ be a genus 0 group. Then, we can determine the unique weakly modular form of weight 0 for Γ which is holomorphic at every cusp but ∞ and has the following form of Fourier expansion:

$$(31) \quad F_{m,\Gamma}(z) := \frac{1}{q_h^m} + \sum_{n=1}^{\infty} a_n q_h^n.$$

Since this function is of weight 0, we can write $F_{m,\Gamma}$ as a polynomial of the hauptmodul for Γ as follows:

$$(32) \quad F_{m,\Gamma}(z) = P_{m,\Gamma}(j_\Gamma(z)),$$

where we call $P_{m,\Gamma}(X)$ the *Hecke type Faber polynomial of degree m for Γ* .

$F_{m,\Gamma}$ and $P_{m,\Gamma}(X)$ are also defined with the *twisted Hecke operator*. (see [ACMS])

For the other cusps, we can also define the functions $F_{m,\Gamma}^\kappa$ and the polynomials $P_{m,\Gamma}^\kappa(X)$, where

$$(33) \quad F_{m,\Gamma}^\kappa|_0\gamma_\kappa(z) := \frac{1}{q_h^m} + \sum_{n=1}^{\infty} a_{n,\kappa} q_h^n,$$

$$(34) \quad F_{m,\Gamma}^\kappa(z) = P_{m,\Gamma}^\kappa(j_\Gamma^\kappa(z)).$$

If $\gamma_\kappa^{-1}\Gamma\gamma_\kappa \subset \Gamma$, then we have $P_{m,\Gamma}^\kappa(X) = P_{m,\Gamma}(X)$ since $j_\Gamma^\kappa(z) = j_\Gamma(\gamma_\kappa^{-1}z)$.

0.3.6. Semimodular form.

Let N be a positive integer. If the function f satisfies that f^n is not a modular form (resp. function) for every integer $n < N$ and f^N is a modular form (resp. function), then we would like to call f the *N th semimodular form (resp. function) for Γ* .

For example, η -function is 24th semimodular form for $\text{SL}_2(\mathbb{Z})$.

Let f and g be meromorphic functions on \mathbb{H} and at every cusp of Γ which satisfy transformation rule of weight k_1 and k_2 for Γ except for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $f(\gamma z) = e^{2\pi i/N}(cz+d)^{k_2}f(z)$ and $g(\gamma z) = e^{2\pi i/N}(cz+d)^{k_2}g(z)$, respectively. Then, f and g are N th semimodular functions for Γ . Furthermore, $f^i g^{N-i}$ and f/g are modular functions for Γ .

0.4. Conjugate of the groups and the space of modular forms.

We denote the space of modular forms of weight k for Γ by $M_k(\Gamma)$, and that of cusp forms by $S_k(\Gamma)$.

Some discrete subgroups of $\text{SL}_2(\mathbb{R})$ are conjugate to each other. Moreover, some spaces of modular forms for such groups are isomorphic to each other. In this section, we introduce two of such cases.

Let Γ and Γ' be discrete subgroups of $\text{SL}_2(\mathbb{R})$ such that they are conjugate to each other.

The first case is when $\Gamma' = V_h^{-1}\Gamma V_h$, where $V_h := \begin{pmatrix} \sqrt{h} & 0 \\ 0 & 1/\sqrt{h} \end{pmatrix}$. Then, it is easy to show that the map

$$M_k(\Gamma) \ni f(z) \mapsto f(hz) \in M_k(\Gamma')$$

is a isomorphism. Furthermore, we have $E_{k,\Gamma}^\infty(hz) = E_{k,\Gamma'}^\infty(z)$. For example, we have $\Gamma_0(3|3) = V_3^{-1}\text{SL}_2(\mathbb{Z})V_3$.

The second case is when $\Gamma' = T_x^{-1}\Gamma T_x$. Then, we can easily show that the map

$$M_k(\Gamma) \ni f(z) \mapsto f(z+x) \in M_k(\Gamma')$$

is a isomorphism. Furthermore, we have $E_{k,\Gamma}^\infty(z+x) = E_{k,\Gamma'}^\infty(z)$. For example, we have $\Gamma_0^*(4) = T_{1/2}^{-1}\Gamma_0(2)T_{1/2}$.

For both above cases, if all of the zeros of $E_{k,\Gamma}^\infty$ lies on the lower acrs of $\partial\mathbb{F}_{0,\Gamma}$, then all of the zeros of $E_{k,\Gamma'}^\infty$ lies on the lower acrs of $\partial\mathbb{F}_{0,\Gamma'}$.

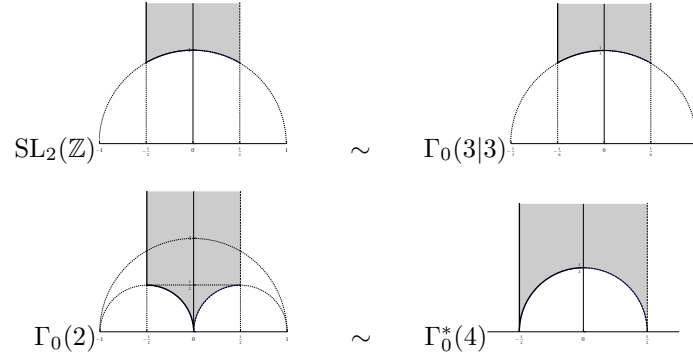


FIGURE 5. Conjugate groups

Remark 0.1. We have

$$(35) \quad \Gamma_0(n|h) + e_1, e_2, \dots, e_m = V_h^{-1}(\Gamma_0(n/h) + e_1, e_2, \dots, e_m)V_h.$$

Thus, in this paper, we observe only the normalizers such that $h = 1$.

1. LEVEL 1

1.1. $\mathbf{SL}_2(\mathbb{Z})$. (see [Se] and [RSD])

Fundamental domain. We have a fundamental domain for $\mathbf{SL}_2(\mathbb{Z})$ as follows:

$$(36) \quad \mathbb{F} = \{|z| \geq 1, -1/2 \leq \operatorname{Re}(z) \leq 0\} \cup \{|z| > 1, 0 < \operatorname{Re}(z) < 1/2\},$$

where $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : e^{i\theta} \rightarrow e^{i(\pi-\theta)}$. By corollary 0.1.1, we have $\mathbf{SL}_2(\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$.

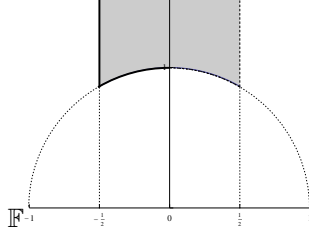


FIGURE 6. $\mathbf{SL}_2(\mathbb{Z})$

Valence formula. The cusp of $\mathbf{SL}_2(\mathbb{Z})$ is only ∞ , and the elliptic points are i and $\rho := e^{2\pi/3}$. Let f be a modular function of weight k for $\mathbf{SL}_2(\mathbb{Z})$, which is not identically zero. We have

$$(37) \quad v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_\rho(f) + \sum_{\substack{p \in \mathbf{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \\ p \neq i, \rho}} v_p(f) = \frac{k}{12}.$$

Furthermore, the stabilizer of the elliptic point i is $\{\pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$, and that of ρ is $\{\pm I, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\}$.

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\mathbf{SL}_2(\mathbb{Z})$:

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k} \quad \text{for } k \geq 4.$$

We also have its Fourier Expansion:

$$(38) \quad E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $q := e^{2\pi iz}$, $\sigma_k(n) := \sum_{d|n} d^k$ which is called *divisor function*, and B_k are *Bernoulli number*.

The space of modular forms. We have $M_k(\mathbf{SL}_2(\mathbb{Z})) = \mathbb{C}E_k \oplus S_k(\mathbf{SL}_2(\mathbb{Z}))$ and $S_k(\mathbf{SL}_2(\mathbb{Z})) = \Delta M_{k-12}(\mathbf{SL}_2(\mathbb{Z}))$ for every even integer $k \geq 4$. Then, we have

$$M_k(\mathbf{SL}_2(\mathbb{Z})) = E_{k-12n}(\mathbb{C}((E_4)^3)^n \oplus \mathbb{C}((E_4)^3)^{n-1} \Delta \oplus \cdots \oplus \mathbb{C}(\Delta)^n),$$

where $n = \dim(M_k(\mathbf{SL}_2(\mathbb{Z}))) - 1 = \lfloor k/12 - (k/4 - \lfloor k/4 \rfloor) \rfloor$. Furthermore, on the zeros of E_{k-12n} , we have $v_\rho(E_4) = 1$, $v_i(E_6) = 1$, $E_8 = (E_4)^2$, $E_{10} = E_4 E_6$, and $E_{14} = (E_4)^2 E_6$.

Hauptmodul. We define the *hauptmodul* of $\mathbf{SL}_2(\mathbb{Z})$:

$$(39) \quad J := (E_4)^3 / \Delta = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

where $v_\infty(J) = -1$ and $v_\rho(J) = 3$. Then, we have

$$(40) \quad J : \partial\mathbb{F} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 1728] \subset \mathbb{R}.$$

2. LEVEL 2

We have $\Gamma_0(2)+ = \Gamma_0^*(2)$ and $\Gamma_0(2)- = \Gamma_0(2)$.

We have $W_2 = \begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$, and denote $\rho_2 := -1/2 + i/2$. We define

$$(41) \quad \begin{aligned} \Delta_2^\infty(z) &:= \eta^{16}(2z)/\eta^8(z), & \Delta_2^0(z) &:= \eta^{16}(z)/\eta^8(2z), \\ \Delta_2(z) &:= \Delta_2^\infty(z)\Delta_2^0(z) = \eta^8(z)\eta^8(2z), \end{aligned}$$

where Δ_2^∞ and Δ_2^0 are modular forms for $\Gamma_0(2)$ of weight 4 such that $v_\infty(\Delta_2^\infty) = v_0(\Delta_2^0) = 1$, and Δ_2 is a cusp form for $\Gamma_0(2)$ and $\Gamma_0^*(2)$ of weight 8. Furthermore, we define

$$(42) \quad E_{2,2}'(z) := 2E_2(2z) - E_2(z),$$

where E_2 is the Eisenstein series for $\text{SL}_2(\mathbb{Z})$, and $E_{2,2}'$ is not a Eisenstein series but a modular form for $\Gamma_0(2)$ of weight 2 such that $v_{\rho_2}(E_{2,2}') = 1$.

2.1. $\Gamma_0^*(2)$. (see [MNS])

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(2)$ as follows:

$$(43) \quad \mathbb{F}_{2+} = \left\{ |z| \geq 1/\sqrt{2}, -1/2 \leq \text{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1/\sqrt{2}, 0 < \text{Re}(z) < 1/2 \right\},$$

where $W_2 : e^{i\theta}/\sqrt{2} \rightarrow e^{i(\pi-\theta)}/\sqrt{2}$. Then, we have

$$(44) \quad \Gamma_0^*(2) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_2 \rangle.$$

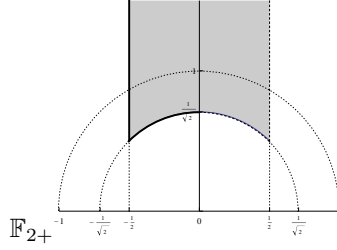


FIGURE 7. $\Gamma_0^*(2)$

Valence formula. The cusp of $\Gamma_0^*(2)$ is ∞ , and the elliptic points are $i/\sqrt{2}$ and $\rho_2 = -1/2 + i/2$. Let f be a modular function of weight k for $\Gamma_0^*(2)$, which is not identically zero. We have

$$(45) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{2}}(f) + \frac{1}{4}v_{\rho_2}(f) + \sum_{\substack{p \in \Gamma_0^*(2) \backslash \mathbb{H} \\ p \neq i/\sqrt{2}, \rho_2}} v_p(f) = \frac{k}{8}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{2}$ is $\{\pm I, \pm W_2\}$, and that of ρ_2 is $\{\pm I, \pm \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_2, \pm \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} W_2\}$.

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0^*(2)$:

$$(46) \quad E_{k,2+}(z) := \frac{2^{k/2}E_k(2z) + E_k(z)}{2^{k/2} + 1} \quad \text{for } k \geq 4.$$

The space of modular forms. We have $M_k(\Gamma_0^*(2)) = \mathbb{C}E_{k,2+} \oplus S_k(\Gamma_0^*(2))$ and $S_k(\Gamma_0^*(2)) = \Delta_2 M_{k-8}(\Gamma_0^*(2))$ for every even integer $k \geq 4$. Then, we have

$$M_k(\Gamma_0^*(2)) = E_{k-8n,2+}(\mathbb{C}((E_{4,2+})^2)^n \oplus \mathbb{C}((E_{4,2+})^2)^{n-1} \Delta_2 \oplus \cdots \oplus \mathbb{C}(\Delta_2)^n),$$

where $n = \dim(M_k(\Gamma_0^*(2))) - 1 = \lfloor k/8 - (k/4 - \lfloor k/4 \rfloor) \rfloor$. Furthermore, on the zeros of $E_{k-8n,2+}$, we have $v_{\rho_2}(E_{4,2+}) = 2$, $v_{i/\sqrt{2}}(E_{6,2+}) = v_{\rho_2}(E_{6,2+}) = 1$, and $E_{10,2+} = E_{4,2+}E_{6,2+}$.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(2)$:

$$(47) \quad J_{2+} := (E_{4,2+})^2 / \Delta_2 = \frac{1}{q} + 104 + 4372q + 96256q^2 + 1240002q^3 + \dots,$$

where $v_\infty(J_{2+}) = -1$ and $v_{\rho_2}(J_{2+}) = 4$. Then, we have

$$(48) \quad J_{2+} : \partial\mathbb{F}_{2+} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 256] \subset \mathbb{R}.$$

2.2. $\Gamma_0(2)$. (see [SJ1])

Fundamental domain. We have a fundamental domain for $\Gamma_0(2)$ as follows:

$$(49) \quad \mathbb{F}_2 = \{|z + 1/2| \geq 1/2, -1/2 \leq \operatorname{Re}(z) \leq 0\} \cup \{|z - 1/2| > 1/2, 0 < \operatorname{Re}(z) < 1/2\},$$

where $\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} : (e^{i\theta} + 1)/2 \rightarrow (e^{i(\pi-\theta)} - 1)/2$. Then we have

$$(50) \quad \Gamma_0(2) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle.$$

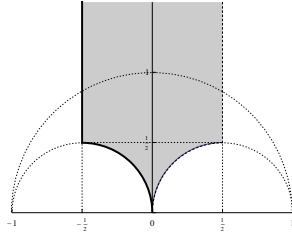


FIGURE 8. $\Gamma_0(2)$

Valence formula. The cusps of $\Gamma_0(2)$ are ∞ and 0, and the elliptic point is ρ_2 . Let f be a modular function of weight k for $\Gamma_0(2)$, which is not identically zero. We have

$$(51) \quad v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_2}(f) + \sum_{\substack{p \in \Gamma_0(2) \setminus \mathbb{H} \\ p \neq \rho_2}} v_p(f) = \frac{k}{4}.$$

Furthermore, the stabilizer of the elliptic point ρ_2 is $\{\pm I, \pm \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}\}$.

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(2)$:

$$(52) \quad E_{k,2}^\infty(z) := \frac{2^k E_k(2z) - E_k(z)}{2^k - 1} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_2$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(2)$:

$$(53) \quad E_{k,2}^0(z) := \frac{-2^{k/2}(E_k(2z) - E_k(z))}{2^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(2) \gamma_0 = \Gamma_0(2)$.

The space of modular forms. We have $M_k(\Gamma_0(2)) = \mathbb{C}E_{k,2}^\infty \oplus \mathbb{C}E_{k,2}^0 \oplus S_k(\Gamma_0(2))$ and $S_k(\Gamma_0(2)) = \Delta_2 M_{k-8}(\Gamma_0(2))$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(2)) = E_{2,2}' M_{4n}(\Gamma_0(2))$ and

$$\begin{aligned} M_{8n}(\Gamma_0(2)) &= \mathbb{C}((E_{4,2}^\infty)^2)^n \oplus \mathbb{C}((E_{4,2}^\infty)^2)^{n-1} \Delta_2 \oplus \dots \oplus \mathbb{C}(E_{4,2}^\infty)^2 (\Delta_2)^{n-1} \\ &\quad \oplus \mathbb{C}((E_{4,2}^0)^2)^n \oplus \mathbb{C}((E_{4,2}^0)^2)^{n-1} \Delta_2 \oplus \dots \oplus \mathbb{C}(E_{4,2}^0)^2 (\Delta_2)^{n-1} \oplus \mathbb{C}(\Delta_2)^n, \\ M_{8n+4}(\Gamma_0(2)) &= E_{4,2}^\infty (\mathbb{C}((E_{4,2}^\infty)^2)^n \oplus \mathbb{C}((E_{4,2}^\infty)^2)^{n-1} \Delta_2 \oplus \dots \oplus \mathbb{C}(\Delta_2)^n) \\ &\quad \oplus E_{4,2}^0 (\mathbb{C}((E_{4,2}^0)^2)^n \oplus \mathbb{C}((E_{4,2}^0)^2)^{n-1} \Delta_2 \oplus \dots \oplus \mathbb{C}(\Delta_2)^n). \end{aligned}$$

Here, since we have $E_{4,2}^\infty = \Delta_2^0$ and $E_{4,2}^0 = \Delta_2^\infty$, we can write

$$M_{4n}(\Gamma_0(2)) = \mathbb{C}(\Delta_2^\infty)^n \oplus \mathbb{C}(\Delta_2^\infty)^{n-1} \Delta_2^0 \oplus \cdots \oplus \mathbb{C}(\Delta_2^0)^n.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(2)$:

$$(54) \quad J_2 := \Delta_2^0 / \Delta_2^\infty (= \eta^{24}(z) / \eta^{24}(2z)) = \frac{1}{q} - 24 + 276q - 2048q^2 + 11202q^3 + \cdots,$$

where $v_\infty(J_2) = -1$ and $v_0(J_2) = 1$. Then, we have

$$(55) \quad J_2 : \partial\mathbb{F}_2 \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [-64, 0] \subset \mathbb{R}.$$

3. LEVEL 3

We have $\Gamma_0(3)+ = \Gamma_0^*(3)$ and $\Gamma_0(3)- = \Gamma_0(3)$.

We have $W_3 = \begin{pmatrix} 0 & -1/\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$, and denote $\rho_3 := -1/2 + i/(2\sqrt{3})$. We define

$$(56) \quad \begin{aligned} \Delta_3^\infty(z) &:= \eta^9(3z)/\eta^3(z), & \Delta_3^0(z) &:= \eta^9(z)/\eta^3(3z), \\ \Delta_3(z) &:= \Delta_3^\infty(z)\Delta_3^0(z) = \eta^6(z)\eta^6(3z), \end{aligned}$$

where Δ_3^∞ and Δ_3^0 are 2nd semimodular forms (cf. Section 0.3.6) for $\Gamma_0(3)$ of weight 3 such that $v_\infty(\Delta_3^\infty) = v_0(\Delta_3^0) = 1$, and Δ_3 is a cusp form for $\Gamma_0(3)$ and a 2nd semimodular form for $\Gamma_0^*(3)$ of weight 6. Furthermore, we define

$$(57) \quad E_{2,3}'(z) := (3E_2(3z) - E_2(z))/2,$$

which is a modular form for $\Gamma_0(3)$ of weight 2 such that $v_{\rho_3}(E_{2,3}') = 2$.

3.1. $\Gamma_0^*(3)$. (see [MNS])

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(3)$ as follows:

$$(58) \quad \mathbb{F}_{3+} = \left\{ |z| \geq 1/\sqrt{3}, -1/2 \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1/\sqrt{3}, 0 < \operatorname{Re}(z) < 1/2 \right\},$$

where $W_3 : e^{i\theta}/\sqrt{3} \rightarrow e^{i(\pi-\theta)}/\sqrt{3}$. Then, we have

$$(59) \quad \Gamma_0^*(3) = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, W_3 \rangle.$$

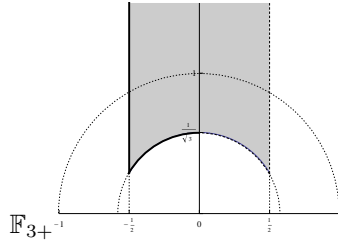


FIGURE 9. $\Gamma_0^*(3)$

Valence formula. The cusp of $\Gamma_0^*(3)$ is ∞ , and the elliptic points are $i/\sqrt{3}$ and $\rho_3 = -1/2 + i/(2\sqrt{3})$. Let f be a modular function of weight k for $\Gamma_0^*(3)$, which is not identically zero. We have

$$(60) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{3}}(f) + \frac{1}{6}v_{\rho_3}(f) + \sum_{\substack{p \in \Gamma_0^*(3) \backslash \mathbb{H} \\ p \neq i/\sqrt{3}, \rho_3}} v_p(f) = \frac{k}{6}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{3}$ is $\{\pm I, \pm W_3\}$, and that of ρ_3 is $\{\pm I, \pm \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} W_3, \pm \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} W_3, \pm \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix} W_3\}$.

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0^*(3)$:

$$(61) \quad E_{k,3+}(z) := \frac{3^{k/2}E_k(3z) + E_k(z)}{3^{k/2} + 1} \quad \text{for } k \geq 4.$$

The space of modular forms. We define the following functions:

$$\begin{aligned}\Delta_{8,3}' &:= (41/1728)((E_{4,3+})^2 - E_{8,3+}), & \Delta_{10,3}' &:= (61/432)(E_{4,3+}E_{6,3+} - E_{10,3+}), \\ \Delta_{12,3}' &:= E_{4,3+}\Delta_{8,3}', & \Delta_{14,3}' &:= E_{4,3+}\Delta_{10,3}'.\end{aligned}$$

Now, we have $M_k(\Gamma_0^*(3)) = \mathbb{C}E_{k,3+} \oplus S_k(\Gamma_0^*(3))$ and $S_k(\Gamma_0^*(3)) = (\mathbb{C}\Delta_{12,3}' \oplus \mathbb{C}(\Delta_3)^2)M_{k-12}(\Gamma_0^*(3))$ for every even integer $k \geq 4$. Then, we have $M_{12n+l} = E_{l,3+}M_{12n}$ for $l = 4, 6$, $M_{12n+l} = E_{l,3+}M_{12n} \oplus \mathbb{C}\Delta_{l,3}'(\Delta_3)^{2n}$ for $l = 8, 10, 14$, and

$$M_{12n}(\Gamma_0^*(3)) = \mathbb{C}(E_{12,3+})^n \oplus \mathbb{C}(E_{12,3+})^{n-1}\Delta_{12,3}' \oplus \mathbb{C}(E_{12,3+})^{n-1}(\Delta_3)^2 \oplus \cdots \oplus \mathbb{C}(\Delta_3)^{2n}.$$

Here, we define $E_{4,3}' := E_{6,3+}/E_{2,3}'$, which is a 2nd semimodular form such that $v_{i/\sqrt{3}}(E_{4,3}') = v_{\rho_3}(E_{4,3}') = 1$. Then, we can write $E_{4,3+} = (E_{2,3}')^2$, $E_{6,3+} = E_{2,3}'E_{4,3}'$, $\Delta_{8,3}' = E_{2,3}'\Delta_3$, $\Delta_{10,3}' = E_{4,3}'\Delta_3$, $\Delta_{12,3}' = (E_{2,3}')^3\Delta_3$, and $\Delta_{14,3}' = (E_{2,3}')^2E_{4,3}'\Delta_3$.

Now, we have

$$M_k(\Gamma_0^*(3)) = E_{\bar{k},3}'(\mathbb{C}((E_{2,3}')^3)^n \oplus \mathbb{C}((E_{2,3}')^3)^{n-1}\Delta_3 \oplus \cdots \oplus \mathbb{C}(\Delta_3)^n),$$

where $n = \dim(M_k(\Gamma_0^*(3))) - 1 = \lfloor k/6 - (k/4 - \lfloor k/4 \rfloor) \rfloor$, and where $E_{\bar{k},3}' := 1, (E_{2,3}')^2E_{4,3}', (E_{2,3}')^2, E_{2,3}'E_{4,3}', E_{2,3}'$, and $E_{4,3}'$, when $k \equiv 0, 2, 4, 6, 8$, and $10 \pmod{12}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(3)$:

$$(62) \quad J_{3+} := (E_{2,3}')^3/\Delta_3 = \frac{1}{q} + 42 + 783q + 8672q^2 + 65367q^3 + \cdots,$$

where $v_\infty(J_{3+}) = -1$ and $v_{\rho_3}(J_{3+}) = 6$. Then, we have

$$(63) \quad J_{3+} : \partial\mathbb{F}_{3+} \setminus \{z \in \mathbb{H}; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 108] \subset \mathbb{R}.$$

3.2. $\Gamma_0(3)$. (see [SJ1])

Fundamental domain. We have a fundamental domain for $\Gamma_0(3)$ as follows:

$$(64) \quad \mathbb{F}_3 = \{|z + 1/3| \geq 1/3, -1/2 \leq \operatorname{Re}(z) \leq 0\} \cup \{|z - 1/3| > 1/3, 0 < \operatorname{Re}(z) < 1/2\},$$

where $\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} : (e^{i\theta} + 1)/3 \rightarrow (e^{i(\pi-\theta)} - 1)/3$. Then, we have

$$(65) \quad \Gamma_0(3) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \rangle.$$

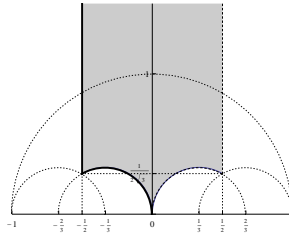


FIGURE 10. $\Gamma_0(3)$

Valence formula. The cusps of $\Gamma_0(3)$ are ∞ and 0 , and the elliptic point is ρ_3 . Let f be a modular function of weight k for $\Gamma_0(3)$, which is not identically zero. We have

$$(66) \quad v_\infty(f) + v_0(f) + \frac{1}{3}v_{\rho_3}(f) + \sum_{\substack{p \in \Gamma_0(3) \backslash \mathbb{H} \\ p \neq \rho_3}} v_p(f) = \frac{k}{3}.$$

Furthermore, the stabilizer of the elliptic point ρ_3 is $\{\pm I, \pm \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}\}$.

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(3)$:

$$(67) \quad E_{k,3}^\infty(z) := \frac{3^k E_k(3z) - E_k(z)}{3^k - 1} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 3n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_3$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(3)$:

$$(68) \quad E_{k,3}^0(z) := \frac{-3^{k/2}(E_k(3z) - E_k(z))}{3^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(3) \gamma_0 = \Gamma_0(3)$.

The space of modular forms. We have $M_k(\Gamma_0(3)) = \mathbb{C}E_{k,3}^\infty \oplus \mathbb{C}E_{k,3}^0 \oplus S_k(\Gamma_0(3))$ and $S_k(\Gamma_0(3)) = \Delta_3 M_{k-6}(\Gamma_0(3))$ for every even integer $k \geq 4$. Then, we have $M_{6n+2}(\Gamma_0(3)) = E_{2,3}' M_{6n}(\Gamma_0(3))$ and

$$\begin{aligned} M_{6n}(\Gamma_0(3)) &= \mathbb{C}(E_{6,3}^\infty)^n \oplus \mathbb{C}(E_{6,3}^\infty)^{n-1} \Delta_3 \oplus \cdots \oplus \mathbb{C}E_{6,3}^\infty (\Delta_3)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{6,3}^0)^n \oplus \mathbb{C}(E_{6,3}^0)^{n-1} \Delta_3 \oplus \cdots \oplus \mathbb{C}E_{6,3}^0 (\Delta_3)^{n-1} \oplus \mathbb{C}(\Delta_3)^n, \\ M_{6n+4}(\Gamma_0(3)) &= E_{4,3}^\infty (\mathbb{C}(E_{6,3}^\infty)^n \oplus \mathbb{C}(E_{6,3}^\infty)^{n-1} \Delta_3 \oplus \cdots \oplus \mathbb{C}(\Delta_3)^n) \\ &\quad \oplus E_{4,3}^0 (\mathbb{C}(E_{6,3}^0)^n \oplus \mathbb{C}(E_{6,3}^0)^{n-1} \Delta_3 \oplus \cdots \oplus \mathbb{C}(\Delta_3)^n). \end{aligned}$$

Here, we define $E_{1,3}' := \sqrt{E_{2,3}'}$, which satisfies $v_{\rho_3}(E_{1,3}') = 1$. Then, we can write $E_{4,3}^\infty = E_{1,3}' \Delta_3^0$ and $(1/27)E_{4,3}^0 = E_{1,3}' \Delta_3^\infty$. Furthermore, we have $E_{6,3}^\infty = (\Delta_3^\infty)^2 + (243/13)\Delta_3^\infty \Delta_3^0$ and $(-13/243)E_{6,3}^0 = \Delta_3^\infty \Delta_3^0 + 39(\Delta_3^0)^2$. Now, we have

$$M_k(\Gamma_0(3)) = E_{k-3n,3}' (\mathbb{C}(\Delta_3^\infty)^n \oplus \mathbb{C}(\Delta_3^\infty)^{n-1} \Delta_3^0 \oplus \cdots \oplus \mathbb{C}(\Delta_3^0)^n),$$

where $n = \dim(M_k(\Gamma_0(3))) - 1 = \lfloor k/3 \rfloor$ and where $E_{0,3}' := 1$.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(3)$:

$$(69) \quad J_3 := \Delta_3^0 / \Delta_3^\infty (= \eta^{12}(z) / \eta^{12}(3z)) = \frac{1}{q} - 12 + 54q - 76q^2 - 243q^3 + \cdots,$$

where $v_\infty(J_3) = -1$ and $v_0(J_3) = 1$. Then, we have

$$(70) \quad J_3 : \partial \mathbb{F}_3 \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-27, 0] \subset \mathbb{R}.$$

4. LEVEL 4

We have $\Gamma_0(4)+ = \Gamma_0(4) + 4 = \Gamma_0^*(4)$ and $\Gamma_0(4)- = \Gamma_0(4)$.

We have $W_4 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}$ and define $W_{4-,2} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$ and $W_{4+,2} = \begin{pmatrix} -1/\sqrt{2} & -3/(2\sqrt{2}) \\ \sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. We define

$$(71) \quad \begin{aligned} \Delta_4^\infty(z) &:= \eta^8(4z)/\eta^4(2z), & \Delta_4^0(z) &:= \eta^8(z)/\eta^4(2z), & \Delta_4^{-1/2}(z) &:= \eta^{20}(2z)/(\eta^8(z)\eta^8(4z)), \\ \Delta_4(z) &:= \Delta_4^\infty(z)\Delta_4^0(z)\Delta_4^{-1/2}(z) = \eta^{12}(2z), \\ \Delta_{4+4}(z) &:= \Delta_4^\infty(z)\Delta_4^0(z)(\Delta_4^{-1/2}(z))^2 = \eta^{32}(2z)/(\eta^8(z)\eta^8(4z)), \end{aligned}$$

where Δ_4^∞ , Δ_4^0 , and $\Delta_4^{-1/2}$ are modular forms for $\Gamma_0(4)$ of weight 2 such that $v_\infty(\Delta_4^\infty) = v_0(\Delta_4^0) = v_{-1/2}(\Delta_4^{-1/2}) = 1$. Then, Δ_4 is a cusp form for $\Gamma_0(4)$ of weight 6, Δ_{4+4} is a cusp form for $\Gamma_0^*(4)$ of weight 8. Furthermore, we define

$$(72) \quad \begin{aligned} E_{2,4}'(z) &:= E_{2,2}'(2z) = 2E_2(4z) - E_2(2z), \\ E_{2,4+4}'(z) &:= 2E_{2,4}'(z) - E_{2,2}'(z) = 4E_2(4z) - 4E_2(2z) + E_2(z), \end{aligned}$$

which are modular forms for $\Gamma_0(4)$, $\Gamma_0^*(4)$ of weight 2, respectively. Then, we have $v_{-1/4+i/4}(E_{2,4}') = 1$ and $v_{i/2}(E_{2,4+4}') = 1$.

4.1. $\Gamma_0^*(4)$. (see [SJ1])

We have $\Gamma_0^*(4) = T_{1/2}^{-1} \Gamma_0(2) T_{1/2}$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(4)$ as follows:

$$(73) \quad \mathbb{F}_{4+4} = \{|z| \geq 1/2, -1/2 \leq \operatorname{Re}(z) \leq 0\} \cup \{|z| > 1/2, 0 < \operatorname{Re}(z) < 1/2\},$$

where $W_4 : e^{i\theta}/2 \rightarrow e^{i(\pi-\theta)}/2$. Then, we have

$$(74) \quad \Gamma_0^*(4) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_4 \rangle.$$

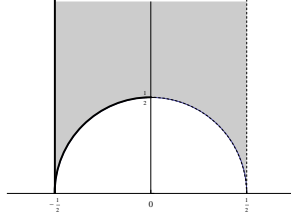


FIGURE 11. $\Gamma_0^*(4)$

Valence formula. The cusps of $\Gamma_0^*(4)$ are ∞ , 0, and $-1/2$, and it has no elliptic point. Let f be a modular function of weight k for $\Gamma_0^*(4)$, which is not identically zero. We have

$$(75) \quad v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/2} + \sum_{\substack{p \in \Gamma_0^*(4) \backslash \mathbb{H} \\ p \neq i/2}} v_p(f) = \frac{k}{4}.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(4)$:

$$(76) \quad E_{k,4+4}^\infty(z) := \frac{2^k E_k(4z) - 2E_k(2z) + E_k(z)}{2^k - 1} \quad \text{for } k \geq 4.$$

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} n+1 & n/2 \\ -2n & -n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{4+,2}$ and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0^*(4)$:

$$(77) \quad E_{k,4+4}^{-1/2}(z) := \frac{-2^{k/2}(2^k E_k(4z) - (2^k + 1)E_k(2z) + E_k(z))}{2^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0^*(4) \gamma_{-1/2} = \Gamma_0^*(4)$.

The space of modular forms. We have

$$\Delta_{4+4}^\infty = \Delta_4^\infty \Delta_4^0, \quad \Delta_{4+4}^{-1/2} = (\Delta_4^{-1/2})^2.$$

Now, we have $M_k(\Gamma_0^*(4)) = \mathbb{C}E_{k,4+4}^\infty \oplus \mathbb{C}E_{k,4}^0 \oplus S_k(\Gamma_0^*(4))$ and $S_k(\Gamma_0^*(4)) = \Delta_{4+4} M_{k-6}(\Gamma_0^*(4))$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0^*(4)) = E_{2,4+4}' M_{4n}(\Gamma_0^*(4))$ and

$$\begin{aligned} M_{8n}(\Gamma_0^*(4)) &= \mathbb{C}((E_{4,4+4}^\infty)^2)^n \oplus \mathbb{C}((E_{4,4+4}^\infty)^2)^{n-1} \Delta_{4+4} \oplus \cdots \oplus \mathbb{C}(E_{4,4+4}^\infty)^2 (\Delta_{4+4})^{n-1} \\ &\quad \oplus \mathbb{C}((E_{4,4+4}^{-1/2})^2)^n \oplus \mathbb{C}((E_{4,4+4}^{-1/2})^2)^{n-1} \Delta_{4+4} \oplus \cdots \oplus \mathbb{C}(E_{4,4+4}^{-1/2})^2 (\Delta_{4+4})^{n-1} \oplus \mathbb{C}(\Delta_{4+4})^n, \\ M_{8n+4}(\Gamma_0^*(4)) &= E_{4,4+4}^\infty (\mathbb{C}((E_{4,4+4}^\infty)^2)^n \oplus \mathbb{C}((E_{4,4+4}^\infty)^2)^{n-1} \Delta_{4+4} \oplus \cdots \oplus \mathbb{C}(\Delta_{4+4})^n) \\ &\quad \oplus E_{4,4+4}^{-1/2} (\mathbb{C}((E_{4,4+4}^{-1/2})^2)^n \oplus \mathbb{C}((E_{4,4+4}^{-1/2})^2)^{n-1} \Delta_{4+4} \oplus \cdots \oplus \mathbb{C}(\Delta_{4+4})^n). \end{aligned}$$

Here, since we have $E_{4,4+4}^\infty = \Delta_{4+4}^{-1/2}$ and $E_{4,4+4}^{-1/2} = \Delta_{4+4}^\infty$, we can write

$$M_{4n}(\Gamma_0^*(4)) = \mathbb{C}(\Delta_{4+4}^\infty)^n \oplus \mathbb{C}(\Delta_{4+4}^\infty)^{n-1} \Delta_{4+4}^{-1/2} \oplus \cdots \oplus \mathbb{C}(\Delta_{4+4}^{-1/2})^n.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(4)$:

$$(78) \quad J_{4+4} := \Delta_{4+4}^{-1/2} / \Delta_{4+4}^\infty (= \eta^{48}(2z) / (\eta^{24}(z)\eta^{24}(4z))) = \frac{1}{q} + 24 + 276q + 2048q^2 + 11202q^3 + \cdots,$$

where $v_\infty(J_{4+4}) = -1$ and $v_{-1/2}(J_{4+4}) = 1$. Then, we have

$$(79) \quad J_{4+4} : \partial \mathbb{F}_{4+4} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 64] \subset \mathbb{R}.$$

4.2. $\Gamma_0(4)$. (see [SJ1])

We have $\Gamma_0(4) = V_2^{-1} \Gamma(2) V_2$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(4)$ as follows:

$$(80) \quad \mathbb{F}_4 = \{ |z + 1/4| \geq 1/4, -1/2 \leq \operatorname{Re}(z) \leq 0 \} \cup \{ |z - 1/4| > 1/4, 0 < \operatorname{Re}(z) < 1/2 \},$$

where $\begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} : (e^{i\theta} + 1)/4 \rightarrow (e^{i(\pi-\theta)} - 1)/4$. Then, we have

$$(81) \quad \Gamma_0(4) = \langle -I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle.$$

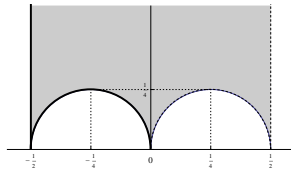


FIGURE 12. $\Gamma_0(4)$

Valence formula. The cusps of $\Gamma_0(4)$ are ∞ , 0 , and $-1/2$, and it has no elliptic point. Let f be a modular function of weight k for $\Gamma_0(4)$, which is not identically zero. We have

$$(82) \quad v_\infty(f) + v_0(f) + v_{-1/2}(f) + \sum_{p \in \Gamma_0(4) \setminus \mathbb{H}} v_p(f) = \frac{k}{2}.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(4)$:

$$(83) \quad E_{k,4}^\infty(z) := \frac{2^k E_k(4z) - E_k(2z)}{2^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^\infty(z) = E_{k,2}^\infty(2z)$.

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 4n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_4$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(4)$:

$$(84) \quad E_{k,4}^0(z) := \frac{-(E_k(2z) - E_k(z))}{2^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^0 = 2^{-k/2} E_{k,2}^0$. We also have $\gamma_0^{-1} \Gamma_0(4) \gamma_0 = \Gamma_0(4)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{4,-,2}$ and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(4)$:

$$(85) \quad E_{k,4}^{-1/2}(z) := \frac{-(2^k E_k(4z) - (2^k + 1)E_k(2z) + E_k(z))}{2^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^{-1/2} = 2^{-k/2} E_{k,2+}^{-1/2}$. We also have $\gamma_{-1/2}^{-1} \Gamma_0(4) \gamma_{-1/2} = \Gamma_0(4)$.

The space of modular forms. We have $E_{2,2}'$ as a modular form for $\Gamma_0(4) \subset \Gamma_0(2)$ of weight 2 such that $v_{-1/2+i/2}(E_{2,2}') = 1$. Moreover, we have $M_2(\Gamma_0(4)) = \mathbb{C}E_{2,2}' \oplus \mathbb{C}E_{2,4}'$.

Now, we have $M_k(\Gamma_0(4)) = \mathbb{C}E_{k,4}^\infty \oplus \mathbb{C}E_{k,4}^0 \oplus \mathbb{C}E_{k,4}^{-1/2} \oplus S_k(\Gamma_0(4))$ and $S_k(\Gamma_0(4)) = \Delta_4 M_{k-6}(\Gamma_0(4))$ for every even integer $k \geq 4$. Then, we have $M_{6n+2}(\Gamma_0(4)) = E_{2,4}' M_{6n}(\Gamma_0(4)) \oplus \mathbb{C}E_{2,2}'(\Delta_4)^n$ and

$$\begin{aligned} M_{6n}(\Gamma_0(4)) &= \mathbb{C}(E_{6,4}^\infty)^n \oplus \mathbb{C}(E_{6,4}^\infty)^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}E_{6,4}^\infty(\Delta_4)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{6,4}^0)^n \oplus \mathbb{C}(E_{6,4}^0)^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}E_{6,4}^0(\Delta_4)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{6,4}^{-1/2})^n \oplus \mathbb{C}(E_{6,4}^{-1/2})^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}E_{6,4}^{-1/2}(\Delta_4)^{n-1} \oplus \mathbb{C}(\Delta_4)^n, \\ M_{6n+4}(\Gamma_0(4)) &= E_{4,4}^\infty(\mathbb{C}(E_{6,4}^\infty)^n \oplus \mathbb{C}(E_{6,4}^\infty)^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}(\Delta_4)^n) \\ &\quad \oplus E_{4,4}^0(\mathbb{C}(E_{6,4}^0)^n \oplus \mathbb{C}(E_{6,4}^0)^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}(\Delta_4)^n) \\ &\quad \oplus E_{4,4}^{-1/2}(\mathbb{C}(E_{6,4}^{-1/2})^n \oplus \mathbb{C}(E_{6,4}^{-1/2})^{n-1} \Delta_4 \oplus \cdots \oplus \mathbb{C}(\Delta_4)^n). \end{aligned}$$

Here, we have $E_{2,2}' = 32\Delta_4^\infty + \Delta_4^0$, $E_{2,4}' = 16\Delta_4^\infty + \Delta_4^0$, $E_{4,4}^\infty = 16\Delta_4^\infty \Delta_4^0 + (\Delta_4^0)^2$, $(1/16)E_{4,4}^0 = 16(\Delta_4^\infty)^2 + \Delta_4^\infty \Delta_4^0$, $(-1/16)E_{4,4}^{-1/2} = \Delta_4^\infty \Delta_4^0$, $E_{6,4}^\infty = 128(\Delta_4^\infty)^2 \Delta_4^0 + 24\Delta_4^\infty (\Delta_4^0)^2 + (\Delta_4^0)^3$, $(-1/8)E_{6,4}^0 = 512(\Delta_4^\infty)^3 + 48(\Delta_4^\infty)^2 \Delta_4^0 + \Delta_4^\infty (\Delta_4^0)^2$, and $(1/8)E_{6,4}^{-1/2} = -16(\Delta_4^\infty)^2 \Delta_4^0 + \Delta_4^\infty (\Delta_4^0)^2$. Now, we can write

$$M_{2n}(\Gamma_0(4)) = \mathbb{C}(\Delta_4^\infty)^n \oplus \mathbb{C}(\Delta_4^\infty)^{n-1} \Delta_4^0 \oplus \cdots \oplus \mathbb{C}(\Delta_4^0)^n.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(4)$:

$$(86) \quad J_4 := \Delta_4^0 / \Delta_4^\infty (= \eta^8(z) / \eta^8(4z)) = \frac{1}{q} - 8 + 20q - 62q^3 + 216q^5 + \cdots,$$

where $v_\infty(J_4) = -1$ and $v_0(J_4) = 1$. Then, we have

$$(87) \quad J_4 : \partial\mathbb{F}_4 \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-16, 0] \subset \mathbb{R}.$$

5. LEVEL 5

We have $\Gamma_0(5)_+ = \Gamma_0^*(5)$ and $\Gamma_0(5)_- = \Gamma_0(5)$.

We have $W_5 = \begin{pmatrix} 0 & -1/\sqrt{5} \\ \sqrt{5} & 0 \end{pmatrix}$, and denote $\rho_{5,1} := -1/2 + i/(2\sqrt{5})$, $\rho_{5,2} := -2/5 + i/5$, and $\rho_{5,3} := 2/5 + i/5$. We define

$$(88) \quad \begin{aligned} \Delta_5^\infty(z) &:= \eta^5(5z)/\eta(z), & \Delta_5^0(z) &:= \eta^5(z)/\eta(5z), \\ \Delta_5(z) &:= \Delta_5^\infty(z)\Delta_5^0(z) = \eta^4(z)\eta^4(5z), \end{aligned}$$

where Δ_5^∞ and Δ_5^0 are 2nd semimodular forms for $\Gamma_0(5)$ of weight 2 such that $v_\infty(\Delta_5^\infty) = v_0(\Delta_5^0) = 1$, and Δ_5 is a cusp form for $\Gamma_0(5)$ and $\Gamma_0^*(5)$ of weight 4. Furthermore, we define

$$(89) \quad E_{2,5}'(z) := (5E_2(5z) - E_2(z))/4,$$

which is a modular form for $\Gamma_0(5)$ and 2nd semimodular form for $\Gamma_0^*(5)$ of weight 2 such that $v_{\rho_{5,2}}(E_{2,5}') = 1$.

5.1. $\Gamma_0^*(5)$. (see [SJ2])

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(5)$ as follows:

$$(90) \quad \mathbb{F}_{5+} = \left\{ |z + 1/2| \geq 1/(2\sqrt{5}), -1/2 \leq \operatorname{Re}(z) < -2/5 \right\} \cup \left\{ |z| \geq 1/\sqrt{5}, -2/5 \leq \operatorname{Re}(z) \leq 0 \right\} \\ \cup \left\{ |z| > 1/\sqrt{5}, 0 < \operatorname{Re}(z) \leq 2/5 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{5}), 2/5 < \operatorname{Re}(z) < 1/2 \right\},$$

where $W_5 : e^{i\theta}/\sqrt{5} \rightarrow e^{i(\pi-\theta)}/\sqrt{5}$ and $\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} W_5 : e^{i\theta}/(2\sqrt{5}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{5}) - 1/2$. Then, we have

$$(91) \quad \Gamma_0^*(5) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_5, \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \rangle.$$

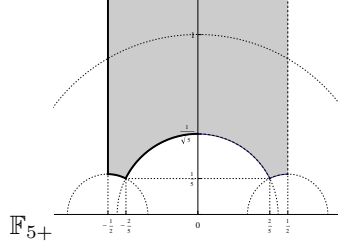


FIGURE 13. $\Gamma_0^*(5)$

Valence formula. The cusp of $\Gamma_0^*(5)$ is ∞ , and the elliptic points are $i/\sqrt{5}$, $\rho_{5,1} = -1/2 + i\sqrt{5}/10$ and $\rho_{5,2} = -2/5 + i/5$. Let f be a modular function of weight k for $\Gamma_0^*(5)$, which is not identically zero. We have

$$(92) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{5}}(f) + \frac{1}{2}v_{\rho_{5,1}}(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \sum_{\substack{p \in \Gamma_0^*(5) \backslash \mathbb{H} \\ p \neq i/\sqrt{5}, \rho_{5,1}, \rho_{5,2}}} v_p(f) = \frac{k}{4}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{5}$ (resp. $\rho_{5,1}$, $\rho_{5,2}$) is $\{\pm I, \pm W_5\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} W_5\}$, $\{\pm I, \pm \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\}$)

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0^*(5)$:

$$(93) \quad E_{k,5+}(z) := \frac{5^{k/2}E_k(5z) + E_k(z)}{5^{k/2} + 1} \quad \text{for } k \geq 4.$$

The space of modular forms. We have $M_k(\Gamma_0^*(5)) = \mathbb{C}E_{k,5+} \oplus S_k(\Gamma_0^*(5))$ and $S_k(\Gamma_0^*(5)) = \Delta_5 M_{k-4}(\Gamma_0^*(5))$ for every even integer $k \geq 4$. Then, we have $M_{4n+6}(\Gamma_0^*(5)) = E_{6,5+} M_{4n}(\Gamma_0^*(5))$ and

$$M_{4n}(\Gamma_0^*(5)) = \mathbb{C}((E_{2,5}')^2)^n \oplus \mathbb{C}((E_{2,5}')^2)^{n-1} \Delta_5 \oplus \cdots \oplus \mathbb{C}(\Delta_5)^n.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(5)$:

$$(94) \quad J_{5+} := (E_{2,5}')^2 / \Delta_5 = \frac{1}{q} + 16 + 134q + 760q^2 + 3345q^3 + \cdots,$$

where $v_\infty(J_{5+}) = -1$ and $v_{\rho_{5,2}}(J_{5+}) = 2$. Then, we have

$$(95) \quad J_{5+} : \partial \mathbb{F}_{5+} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [22 - 10\sqrt{5}, 22 + 10\sqrt{5}] \subset \mathbb{R}.$$

5.2. $\Gamma_0(5)$. (see [SJ1])

Fundamental domain. We have a fundamental domain for $\Gamma_0(5)$ as follows:

$$(96) \quad \begin{aligned} \mathbb{F}_5 = & \{ |z + 2/5| \geq 1/5, -1/2 \leq \operatorname{Re}(z) \leq -2/5 \} \cup \{ |z + 2/5| > 1/5, -2/5 < \operatorname{Re}(z) < -3/10 \} \\ & \cup \{ |z + 1/5| \geq 1/5, -3/10 \leq \operatorname{Re}(z) \leq 0 \} \cup \{ |z - 1/5| > 1/5, 0 < \operatorname{Re}(z) < 3/10 \} \\ & \cup \{ |z - 2/5| \geq 1/5, 3/10 \leq \operatorname{Re}(z) \leq 2/5 \} \cup \{ |z - 2/5| > 1/5, 2/5 < \operatorname{Re}(z) < 1/2 \}, \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix} : (e^{i\theta} + 1)/5 \rightarrow (e^{i(\pi-\theta)} - 1)/5$, $\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} : (e^{i\theta} + 2)/5 \rightarrow (e^{i(\pi-\theta)} + 2)/5$, and $\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} : (e^{i\theta} - 2)/5 \rightarrow (e^{i(\pi-\theta)} - 2)/5$. Then, we have

$$(97) \quad \Gamma_0(5) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \rangle.$$

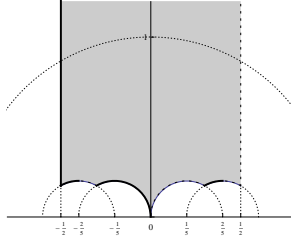


FIGURE 14. $\Gamma_0(5)$

Valence formula. The cusps of $\Gamma_0(5)$ are ∞ and 0 , and the elliptic points are $\rho_{5,2}$ and $\rho_{5,3} = 2/5 + i/5$. Let f be a modular function of weight k for $\Gamma_0(5)$, which is not identically zero. We have

$$(98) \quad v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_{5,2}}(f) + \frac{1}{2}v_{\rho_{5,3}}(f) + \sum_{\substack{p \in \Gamma_0(5) \setminus \mathbb{H} \\ p \neq \rho_{5,2}, \rho_{5,3}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{5,2}$ (resp. $\rho_{5,3}$) is $\{\pm I, \pm \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(5)$:

$$(99) \quad E_{k,5}^\infty(z) := \frac{5^k E_k(5z) - E_k(z)}{5^k - 1} \quad \text{for } k \geq 4.$$

For the cusp 0 . We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 5n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_5$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(5)$:

$$(100) \quad E_{k,5}^0(z) := \frac{-5^{k/2}(E_k(5z) - E_k(z))}{5^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(5) \gamma_0 = \Gamma_0(5)$.

The space of modular forms. We have $M_k(\Gamma_0(5)) = \mathbb{C}E_{k,5}^\infty \oplus \mathbb{C}E_{k,5}^0 \oplus S_k(\Gamma_0(5))$ and $S_k(\Gamma_0(5)) = \Delta_5 M_{k-4}(\Gamma_0(5))$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(5)) = E_{2,5}' M_{4n}(\Gamma_0(5))$ and

$$M_{4n}(\Gamma_0(5)) = \mathbb{C}(E_{4,5}^\infty)^n \oplus \mathbb{C}(E_{4,5}^\infty)^{n-1} \Delta_5 \oplus \cdots \oplus \mathbb{C}E_{4,5}^\infty (\Delta_5)^{n-1} \\ \oplus \mathbb{C}(E_{4,5}^0)^n \oplus \mathbb{C}(E_{4,5}^0)^{n-1} \Delta_5 \oplus \cdots \oplus \mathbb{C}E_{4,5}^0 (\Delta_5)^{n-1} \oplus \mathbb{C}(\Delta_5)^n.$$

Here, we have $E_{4,5}^\infty = (125/13)\Delta_5^\infty \Delta_5^0 + (\Delta_5^0)^2$ and $(13/125)E_{4,5}^0 = 13(\Delta_5^\infty)^2 + \Delta_5^\infty \Delta_5^0$, then we can write

$$M_{4n}(\Gamma_0(5)) = \mathbb{C}(\Delta_5^\infty)^{2n} \oplus \mathbb{C}(\Delta_5^\infty)^{2n-1} \Delta_5^0 \oplus \cdots \oplus \mathbb{C}(\Delta_5^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(5)$:

$$(101) \quad J_5 := \Delta_5^0 / \Delta_5^\infty (= \eta^6(z) / \eta^6(5z)) = \frac{1}{q} - 6 + 9q + 10q^2 - 30q^3 + \cdots,$$

where $v_\infty(J_5) = -1$ and $v_0(J_5) = 1$. Then, we have

$$(102) \quad J_5 : \quad \begin{aligned} \{|z + 1/5| = 1/5, -3/10 \leq \operatorname{Re}(z) \leq 0\} &\rightarrow [-5 - 2\sqrt{5}, 0] \subset \mathbb{R}, \\ \{|z + 2/5| = 1/5, -1/2 \leq \operatorname{Re}(z) \leq -2/5\} &\rightarrow \{-11 \leq \operatorname{Re}(z) \leq -5 - 2\sqrt{5}, 0 \leq \operatorname{Im}(z) \leq 2\}, \\ \{|z - 2/5| = 1/5, 3/10 \leq \operatorname{Re}(z) \leq 2/5\} &\rightarrow \{-11 \leq \operatorname{Re}(z) \leq -5 - 2\sqrt{5}, -2 \leq \operatorname{Im}(z) \leq 0\}. \end{aligned}$$

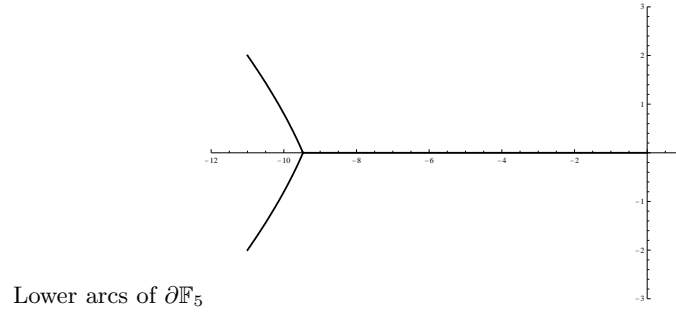


FIGURE 15. Image by J_5

6. LEVEL 6

We have $\Gamma_0(6)+$, $\Gamma_0(6) + 6 = \Gamma_0^*(6)$, $\Gamma_0(6) + 3$, $\Gamma_0(6) + 2$, and $\Gamma_0(6)- = \Gamma_0(6)$.

We have $W_6 = \begin{pmatrix} 0 & -1/\sqrt{6} \\ \sqrt{6} & 0 \end{pmatrix}$, $W_{6,2} := \begin{pmatrix} -\sqrt{2} & -1/\sqrt{2} \\ 3\sqrt{2} & \sqrt{2} \end{pmatrix}$, and $W_{6,3} := \begin{pmatrix} -\sqrt{3} & -2/\sqrt{3} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix}$, and we denote $\rho_{6,1} := \rho_3 = -1/2 + i/(2\sqrt{3})$, $\rho_{6,2} := -1/3 + i/(3\sqrt{2})$, $\rho_{6,3} := -2/5 + i/(5\sqrt{6})$, $\rho_{6,4} := -1/4 + i/(4\sqrt{3})$, and $\rho_{6,5} := 1/3 + i/(3\sqrt{2})$. We define

$$(103) \quad \begin{aligned} \Delta_6^\infty(z) &:= \eta(z)\eta^{-2}(2z)\eta^{-3}(3z)\eta^6(6z), & \Delta_6^0(z) &:= \eta^6(z)\eta^{-3}(2z)\eta^{-2}(3z)\eta(6z), \\ \Delta_6^{-1/2}(z) &:= \eta^{-3}(z)\eta^6(2z)\eta(3z)\eta^{-2}(6z), & \Delta_6^{-1/3}(z) &:= \eta^{-2}(z)\eta(2z)\eta^6(3z)\eta^{-3}(6z), \\ \Delta_6(z) &:= \Delta_6^\infty(z)\Delta_6^0(z)\Delta_6^{-1/3}(z)\Delta_6^{-1/2}(z) = \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z), \end{aligned}$$

where Δ_6^∞ , Δ_6^0 , $\Delta_6^{-1/2}$, and $\Delta_6^{-1/3}$ are 2nd semimodular forms for $\Gamma_0(6)$ of weight 1 such that $v_\infty(\Delta_6^\infty) = v_0(\Delta_6^0) = v_{-1/2}(\Delta_6^{-1/2}) = v_{-1/3}(\Delta_6^{-1/3}) = 1$, and Δ_6 is a cusp form for $\Gamma_0(6)$ of weight 4. Furthermore, we define

$$(104) \quad \begin{aligned} E_{2,6+6}'(z) &:= (6E_2(6z) - 3E_2(3z) - 2E_2(2z) + E_2(z))/2, \\ E_{2,6+3}'(z) &:= (6E_2(6z) - 3E_2(3z) + 2E_2(2z) - E_2(z))/4, \\ E_{2,6+2}'(z) &:= (6E_2(6z) + 3E_2(3z) - 2E_2(2z) - E_2(z))/6, \end{aligned}$$

which are modular forms for $\Gamma_0(6)$ of weight 2, and we have $v_{i/\sqrt{6}}(E_{2,6+6}') = v_{\rho_{6,3}}(E_{2,6+6}') = 1$, $v_{\rho_{6,1}}(E_{2,6+3}') = v_{\rho_{6,4}}(E_{2,6+3}') = 1$, and $v_{\rho_{6,2}}(E_{2,6+2}') = v_{\rho_{6,5}}(E_{2,6+2}') = 1$.

6.1. $\Gamma_0(6)+$.

We have

$$\Gamma_0(6)+ = \Gamma_0(6) + 2, 3, 6 = \Gamma_0(6) \cup \Gamma_0(6)W_{6,2} \cup \Gamma_0(6)W_{6,3} \cup \Gamma_0(6)W_6.$$

Fundamental domain. We have a fundamental domain for $\Gamma_0(6)+$ as follows:

$$(105) \quad \begin{aligned} \mathbb{F}_{6+} &= \left\{ |z + 1/2| \geq 1/(2\sqrt{3}), -1/2 \leq \operatorname{Re}(z) < -1/3 \right\} \cup \left\{ |z| \geq 1/\sqrt{6}, -1/3 \leq \operatorname{Re}(z) \leq 0 \right\} \\ &\quad \cup \left\{ |z| > 1/\sqrt{6}, 0 < \operatorname{Re}(z) \leq 1/3 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{3}), 1/3 < \operatorname{Re}(z) < 1/2 \right\}, \end{aligned}$$

where, $W_6 : e^{i\theta}/\sqrt{6} \rightarrow e^{i(\pi-\theta)}/\sqrt{6}$ and $\begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} W_{6,3} : e^{i\theta}/(2\sqrt{3}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{3}) - 1/2$. Then, we have

$$(106) \quad \Gamma_0(6)+ = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_6, W_{6,3} \rangle.$$

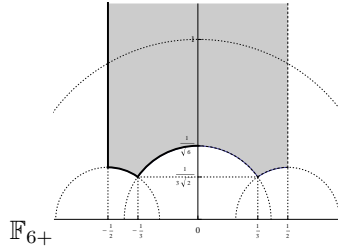


FIGURE 16. $\Gamma_0(6)+$

Valence formula. The cusp of $\Gamma_0(6)+$ is ∞ , and the elliptic points are $i/\sqrt{6}$, $\rho_{6,1} = -1/2 + i/(2\sqrt{3})$, and $\rho_{6,2} = -1/3 + i/(3\sqrt{2})$. Let f be a modular function of weight k for $\Gamma_0(6)+$, which is not identically zero. We have

$$(107) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{6}}(f) + \frac{1}{2}v_{\rho_{6,1}}(f) + \frac{1}{2}v_{\rho_{6,2}}(f) + \sum_{\substack{p \in \Gamma_0(6)+ \setminus \mathbb{H} \\ p \neq i/\sqrt{6}, \rho_{6,1}, \rho_{6,2}}} v_p(f) = \frac{k}{4}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{6}$ (resp. $\rho_{6,1}$, $\rho_{6,2}$) is $\{\pm I, \pm W_6\}$ (resp. $\{\pm I, \pm W_{6,3}\}$, $\{\pm I, \pm W_{6,2}\}$)

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0(6)+$:

$$(108) \quad E_{k,6+}(z) := \frac{6^{k/2}E_k(6z) + 3^{k/2}E_k(3z) + 2^{k/2}E_k(2z) + E_k(z)}{(3^{k/2} + 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

The space of modular forms. We have $M_k(\Gamma_0(6)+) = \mathbb{C}E_{k,6+} \oplus S_k(\Gamma_0(6)+)$ and $S_k(\Gamma_0(6)+) = \Delta_6 M_{k-4}(\Gamma_0(6)+)$ for every even integer $k \geq 4$. Then, we have $M_{4n+6}(\Gamma_0(6)+) = E_{6,6+} M_{4n}(\Gamma_0(6)+)$ and

$$M_{4n}(\Gamma_0(6)+) = \mathbb{C}((E_{2,6+2}')^2)^n \oplus \mathbb{C}((E_{2,6+2}')^2)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}(\Delta_6)^n.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(6)+$:

$$(109) \quad J_{6+} := (E_{2,6+2}')^2 / \Delta_6 = \frac{1}{q} + 10 + 79q + 352q^2 + 1431q^3 + \cdots,$$

where $v_\infty(J_{6+}) = -1$ and $v_{\rho_{6,2}}(J_{6+}) = 2$. Then, we have

$$(110) \quad J_{6+} : \partial\mathbb{F}_{6+} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-4, 32] \subset \mathbb{R}.$$

6.2. $\Gamma_0(6) + 6 = \Gamma_0^*(6)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(6)$ as follows:

$$(111) \quad \mathbb{F}_{6+6} = \{|z + 5/12| \geq 1/12, -1/2 \leq \operatorname{Re}(z) < -2/5\} \cup \{|z| \geq 1/\sqrt{6}, -2/5 \leq \operatorname{Re}(z) \leq 0\} \\ \cup \{|z| > 1/\sqrt{6}, 0 < \operatorname{Re}(z) \leq 2/5\} \cup \{|z - 5/12| > 1/12, 2/5 < \operatorname{Re}(z) < 1/2\},$$

where $W_6 : e^{i\theta}/\sqrt{6} \rightarrow e^{i(\pi-\theta)}/\sqrt{6}$ and $\begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(112) \quad \Gamma_0^*(6) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_6, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix} \rangle.$$

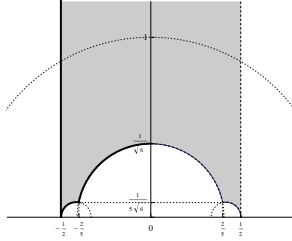


FIGURE 17. $\Gamma_0^*(6)$

Valence formula. The cusps of $\Gamma_0^*(6)$ are ∞ and $-1/2$, and the elliptic points are $i/\sqrt{6}$ and $\rho_{6,3} = -2/5 + i/(5\sqrt{6})$. Let f be a modular function of weight k for $\Gamma_0^*(6)$, which is not identically zero. We have

$$(113) \quad v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/\sqrt{6}}(f) + \frac{1}{2}v_{\rho_{6,3}}(f) + \sum_{\substack{p \in \Gamma_0^*(6) \setminus \mathbb{H} \\ p \neq i/\sqrt{6}, \rho_{6,3}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{6}$ (resp. $\rho_{6,3}$) is $\{\pm I, \pm W_6\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 5 & -2 \\ -12 & 5 \end{pmatrix} W_6\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(6)$:

$$(114) \quad E_{k,6+6}^\infty(z) := \frac{(6^{k/2} + 1)(6^{k/2}E_k(6z) + E_k(z)) - (3^{k/2} + 2^{k/2})(3^{k/2}E_k(3z) + 2^{k/2}E_k(2z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} 6n+1 & 3n \\ -12n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{6,3}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0^*(6)$:

$$(115) \quad E_{k,6+6}^{-1/2}(z) := \frac{-(3^{k/2} + 2^{k/2})(6^{k/2}E_k(6z) + E_k(z)) + (6^{k/2} + 1)(3^{k/2}E_k(3z) + 2^{k/2}E_k(2z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0^*(6) \gamma_{-1/2} = \Gamma_0^*(6)$.

The space of modular forms. We define

$$\Delta_{6+6}^\infty := \Delta_6^\infty \Delta_6^0, \quad \Delta_{6+6}^{-1/2} := \Delta_6^{-1/2} \Delta_6^{-1/3},$$

which are 2nd semimodular forms for $\Gamma_0^*(6)$ of weight 2.

Now, we have $M_k(\Gamma_0^*(6)) = \mathbb{C}E_{k,6+6}^\infty \oplus \mathbb{C}E_{k,6+6}^{-1/2} \oplus S_k(\Gamma_0^*(6))$ and $S_k(\Gamma_0^*(6)) = \Delta_6 M_{k-4}(\Gamma_0^*(6))$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0^*(6)) = E_{2,6+6}' M_{4n}(\Gamma_0^*(6))$ and

$$\begin{aligned} M_{4n}(\Gamma_0^*(6)) &= \mathbb{C}(E_{4,6+6}^\infty)^n \oplus \mathbb{C}(E_{4,6+6}^\infty)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+6}^\infty (\Delta_6)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,6+6}^{-1/2})^n \oplus \mathbb{C}(E_{4,6+6}^{-1/2})^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+6}^{-1/2} (\Delta_6)^{n-1} \oplus \mathbb{C}(\Delta_6)^n. \end{aligned}$$

Here, we have $E_{4,6+6}^\infty = (-13/5)\Delta_{6+6}^\infty \Delta_{6+6}^{-1/2} + (\Delta_{6+6}^{-1/2})^2$ and $(-5/13)E_{4,6+6}^{-1/2} = (-5/13)(\Delta_{6+6}^\infty)^2 + \Delta_{6+6}^\infty \Delta_{6+6}^{-1/2}$, then we can write

$$M_{4n}(\Gamma_0^*(6)) = \mathbb{C}(\Delta_{6+6}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{6+6}^\infty)^{2n-1} \Delta_{6+6}^{-1/2} \oplus \cdots \oplus \mathbb{C}(\Delta_{6+6}^{-1/2})^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(6)$:

$$(116) \quad J_{6+6} := \Delta_{6+6}^{-1/2} / \Delta_{6+6}^\infty (= \eta^5(z)\eta^{-1}(2z)\eta(3z)\eta^{-5}(6z)) = \frac{1}{q} + 12 + 78q + 364q^2 + 1365q^3 + \cdots,$$

where $v_\infty(J_{6+6}) = -1$ and $v_{-1/2}(J_{6+6}) = 1$. Then, we have

$$(117) \quad J_{6+6} : \partial\mathbb{F}_{6+6} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 17 + 12\sqrt{2}] \subset \mathbb{R}.$$

6.3. $\Gamma_0(6) + 3$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(6) + 3$ as follows:

$$(118) \quad \mathbb{F}_{6+3} = \left\{ |z + 1/2| \geq 1/(2\sqrt{3}), -1/2 \leq \operatorname{Re}(z) < -1/4 \right\} \cup \left\{ |z + 1/6| \geq 1/6, -1/4 \leq \operatorname{Re}(z) \leq 0 \right\} \\ \cup \left\{ |z - 1/6| > 1/6, 0 < \operatorname{Re}(z) \leq 1/4 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{3}), 1/4 < \operatorname{Re}(z) < 1/2 \right\},$$

where $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} : (e^{i\theta} + 1)/6 \rightarrow (e^{i(\pi-\theta)} - 1)/6$ and $\begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} W_{6,3} : e^{i\theta}/(2\sqrt{3}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{3}) - 1/2$. Then, we have

$$(119) \quad \Gamma_0(6) + 3 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, W_{6,3} \rangle.$$

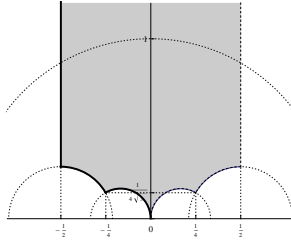


FIGURE 18. $\Gamma_0(6) + 3$

Valence formula. The cusps of $\Gamma_0(6)+3$ are ∞ and 0, and the elliptic points are $\rho_{6,1} = -1/2 + i/(2\sqrt{3})$ and $\rho_{6,4} = -1/4 + i/(4\sqrt{3})$. Let f be a modular function of weight k for $\Gamma_0(6)+3$, which is not identically zero. We have

$$(120) \quad v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_{6,1}}(f) + \frac{1}{2}v_{\rho_{6,4}}(f) + \sum_{\substack{p \in \Gamma_0(6)+3 \setminus \mathbb{H} \\ p \neq \rho_{6,1}, \rho_{6,4}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{6,1}$ (resp. $\rho_{6,4}$) is $\{\pm I, \pm W_{6,3}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} -1 & -1 \\ 6 & 5 \end{pmatrix} W_{6,3}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(6)+3$:

$$(121) \quad E_{k,6+3}^\infty(z) := \frac{2^k 3^{k/2} E_k(6z) - 3^{k/2} E_k(3z) + 2^k E_k(2z) - E_k(z)}{(3^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 6n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_6$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(6)+3$:

$$(122) \quad E_{k,6+3}^0(z) := \frac{-2^{k/2}(3^{k/2} E_k(6z) - 3^{k/2} E_k(3z) + E_k(2z) - E_k(z))}{(3^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1}(\Gamma_0(6)+3)\gamma_0 = \Gamma_0(6)+3$.

The space of modular forms. We define

$$\Delta_{6+3}^\infty := \Delta_6^\infty \Delta_6^{-1/2}, \quad \Delta_{6+3}^0 := \Delta_6^0 \Delta_6^{-1/3},$$

which are 2nd semimodular forms for $\Gamma_0(6)+3$ of weight 2.

Now, we have $M_k(\Gamma_0(6)+3) = \mathbb{C}E_{k,6+3}^\infty \oplus \mathbb{C}E_{k,6+3}^0 \oplus S_k(\Gamma_0(6)+3)$ and $S_k(\Gamma_0(6)+3) = \Delta_6 M_{k-4}(\Gamma_0(6)+3)$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(6)+3) = E_{2,6+3}' M_{4n}(\Gamma_0(6)+3)$ and

$$\begin{aligned} M_{4n}(\Gamma_0(6)+3) &= \mathbb{C}(E_{4,6+3}^\infty)^n \oplus \mathbb{C}(E_{4,6+3}^\infty)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+3}^\infty (\Delta_6)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,6+3}^0)^n \oplus \mathbb{C}(E_{4,6+3}^0)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+3}^0 (\Delta_6)^{n-1} \oplus \mathbb{C}(\Delta_6)^n. \end{aligned}$$

Here, we have $E_{4,6+3}^\infty = (32/5)\Delta_{6+3}^\infty \Delta_{6+3}^0 + (\Delta_{6+3}^0)^2$ and $(5/32)E_{4,6+3}^0 = 10(\Delta_{6+3}^\infty)^2 + \Delta_{6+3}^\infty \Delta_{6+3}^0$, then we can write

$$M_{4n}(\Gamma_0(6)+3) = \mathbb{C}(\Delta_{6+3}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{6+3}^\infty)^{2n-1} \Delta_{6+3}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{6+3}^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(6)+3$:

$$(123) \quad J_{6+3} := \Delta_{6+3}^0 / \Delta_{6+3}^\infty (= \eta^6(z)\eta^{-6}(2z)\eta^6(3z)\eta^{-6}(6z)) = \frac{1}{q} - 6 + 15q - 32q^2 + 87q^3 - \cdots,$$

where $v_\infty(J_{6+3}) = -1$ and $v_0(J_{6+3}) = 1$. Then, we have

$$(124) \quad J_{6+3} : \partial\mathbb{F}_{6+3} \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [-16, 0] \subset \mathbb{R}.$$

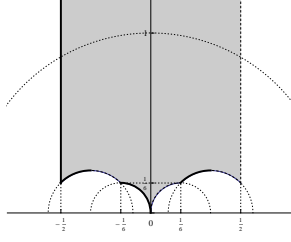
6.4. $\Gamma_0(6)+2$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(6)+2$ as follows:

$$(125) \quad \begin{aligned} \mathbb{F}_{6+2} &= \left\{ |z+1/3| \geq 1/(3\sqrt{2}), -1/2 \leq Re(z) \leq -1/3 \right\} \cup \left\{ |z+1/3| > 1/(3\sqrt{2}), -1/3 < Re(z) < -1/6 \right\} \\ &\cup \left\{ |z+1/6| \geq 1/6, 0 \leq Re(z) \leq 0 \right\} \cup \left\{ |z-1/6| > 1/6, 0 < Re(z) < 1/6 \right\} \\ &\cup \left\{ |z-1/3| \geq 1/(3\sqrt{2}), 1/6 \leq Re(z) \leq 1/3 \right\} \cup \left\{ |z-1/3| > 1/(3\sqrt{2}), 1/3 < Re(z) < 1/2 \right\}, \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} : (e^{i\theta} + 1)/6 \rightarrow (e^{i(\pi-\theta)} - 1)/6, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix} W_{6,3} : e^{i\theta}/(3\sqrt{2}) + 1/3 \rightarrow e^{i(\pi-\theta)}/(3\sqrt{2}) + 1/3$, and $W_{6,3} : e^{i\theta}/(3\sqrt{2}) - 1/3 \rightarrow e^{i(\pi-\theta)}/(3\sqrt{2}) - 1/3$. Then, we have

$$(126) \quad \Gamma_0(6)+2 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, W_{6,2} \rangle.$$

FIGURE 19. $\Gamma_0(6) + 2$

Valence formula. The cusps of $\Gamma_0(6) + 2$ are ∞ and 0, and the elliptic points are $\rho_{6,2} = -1/3 + i/(3\sqrt{2})$ and $\rho_{6,5} = 1/3 + i/(3\sqrt{2})$. Let f be a modular function of weight k for $\Gamma_0(6) + 2$, which is not identically zero. We have

$$(127) \quad v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_{6,2}}(f) + \frac{1}{2}v_{\rho_{6,5}}(f) + \sum_{\substack{p \in \Gamma_0(6)+2 \setminus \mathbb{H} \\ p \neq \rho_{6,2}, \rho_{6,5}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{6,2}$ (resp. $\rho_{6,5}$) is $\{\pm I, \pm W_{6,2}\}$ (resp. $\{\pm I, \pm (\begin{smallmatrix} 5 & 2 \\ 12 & 5 \end{smallmatrix}) W_{6,2}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm (\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}) ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(6) + 2$:

$$(128) \quad E_{k,6+2}^\infty(z) := \frac{2^{k/2}3^k E_k(6z) + 3^k E_k(3z) - 2^{k/2} E_k(2z) - E_k(z)}{(3^k - 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_6$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(6) + 2$:

$$(129) \quad E_{k,6+2}^0(z) := \frac{-3^{k/2}(2^{k/2} E_k(6z) + E_k(3z) - 2^{k/2} E_k(2z) - E_k(z))}{(3^k - 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1}(\Gamma_0(6) + 2)\gamma_0 = \Gamma_0(6) + 2$.

The space of modular forms. We define

$$\Delta_{6+2}^\infty := \Delta_6^\infty \Delta_6^{-1/3}, \quad \Delta_{6+2}^0 := \Delta_6^0 \Delta_6^{-1/2},$$

which are 2nd semimodular forms for $\Gamma_0(6) + 2$ of weight 2.

Now, we have $M_k(\Gamma_0(6) + 2) = \mathbb{C}E_{k,6+2}^\infty \oplus \mathbb{C}E_{k,6+2}^0 \oplus S_k(\Gamma_0(6) + 2)$ and $S_k(\Gamma_0(6) + 2) = \Delta_6 M_{k-4}(\Gamma_0(6) + 2)$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(6) + 2) = E_{2,6+2}' M_{4n}(\Gamma_0(6) + 2)$ and

$$\begin{aligned} M_{4n}(\Gamma_0(6) + 2) &= \mathbb{C}(E_{4,6+2}^\infty)^n \oplus \mathbb{C}(E_{4,6+2}^\infty)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+2}^\infty (\Delta_6)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,6+2}^0)^n \oplus \mathbb{C}(E_{4,6+2}^0)^{n-1} \Delta_6 \oplus \cdots \oplus \mathbb{C}E_{4,6+2}^0 (\Delta_6)^{n-1} \oplus \mathbb{C}(\Delta_6)^n. \end{aligned}$$

Here, we have $E_{4,6+2}^\infty = (27/5)\Delta_{6+2}^\infty \Delta_{6+2}^0 + (\Delta_{6+2}^0)^2$ and $(5/27)E_{4,6+2}^0 = 7(\Delta_{6+2}^\infty)^2 + \Delta_{6+2}^\infty \Delta_{6+2}^0$, then we can write

$$M_{4n}(\Gamma_0(6) + 2) = \mathbb{C}(\Delta_{6+2}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{6+2}^\infty)^{2n-1} \Delta_{6+2}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{6+2}^0)^{2n}.$$

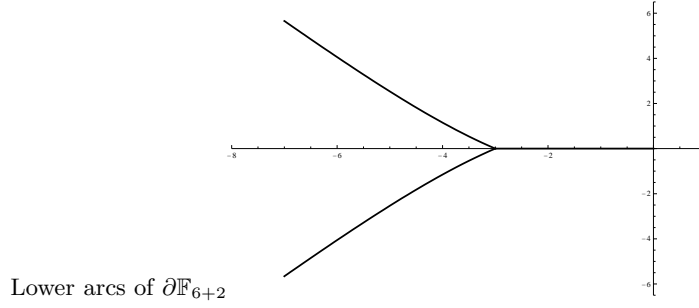
Hauptmodul. We define the *hauptmodul* of $\Gamma_0(6) + 2$:

$$(130) \quad J_{6+2} := \Delta_{6+2}^0 / \Delta_{6+2}^\infty (= \eta^4(z)\eta^4(2z)\eta^{-4}(3z)\eta^{-4}(6z)) = \frac{1}{q} - 4 - 2q + 28q^2 - 27q^3 - \cdots,$$

where $v_\infty(J_{6+2}) = -1$ and $v_0(J_{6+2}) = 1$. Then, we have

$$(131) \quad \begin{aligned} J_{6+2} : \quad & \{|z + 1/6| = 1/6, -1/6 \leq \operatorname{Re}(z) \leq 0\} \rightarrow [-3, 0] \subset \mathbb{R}, \\ & \{|z + 1/3| = 1/(3\sqrt{2}), -1/2 \leq \operatorname{Re}(z) \leq -1/3\} \rightarrow \{-7 \leq \operatorname{Re}(z) \leq -3, 0 \leq \operatorname{Im}(z) \leq 4\sqrt{2}\}, \\ & \{|z - 1/3| = 1/(3\sqrt{2}), 1/6 \leq \operatorname{Re}(z) \leq 1/3\} \rightarrow \{-7 \leq \operatorname{Re}(z) \leq -3, -4\sqrt{2} \leq \operatorname{Im}(z) \leq 0\}. \end{aligned}$$

Thus, J_{6+2} does not take real value on some arcs of $\partial\mathbb{F}_{6+2}$.

FIGURE 20. Image by J_{6+2}

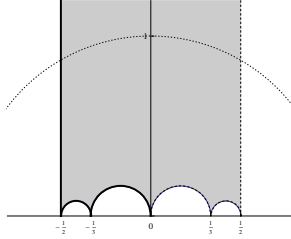
6.5. $\Gamma_0(6)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(6)$ as follows:

$$(132) \quad \mathbb{F}_6 = \{|z + 5/12| \geq 1/12, -1/2 \leq \operatorname{Re}(z) < -1/3\} \cup \{|z + 1/6| \geq 1/6, -1/3 \leq \operatorname{Re}(z) \leq 0\} \\ \cup \{|z - 1/6| > 1/6, 0 < \operatorname{Re}(z) \leq 1/3\} \cup \{|z - 5/12| > 1/12, 1/3 < \operatorname{Re}(z) < 1/2\},$$

where $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} : (e^{i\theta} + 1)/6 \rightarrow (e^{i(\pi-\theta)} - 1)/6$ and $\begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(133) \quad \Gamma_0(6) = \langle -I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix} \rangle.$$

FIGURE 21. $\Gamma_0(6)$

Valence formula. The cusps of $\Gamma_0(6)$ are ∞ , 0 , $-1/2$, and $-1/3$. Let f be a modular function of weight k for $\Gamma_0(6)$, which is not identically zero. We have

$$(134) \quad v_\infty(f) + v_0(f) + v_{-1/2}(f) + v_{-1/3}(f) + \sum_{p \in \Gamma_0(6) \backslash \mathbb{H}} v_p(f) = k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(6)$:

$$(135) \quad E_{k,6}^\infty(z) := \frac{6^k E_k(6z) - 3^k E_k(3z) - 2^k E_k(2z) + E_k(z)}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp 0 . We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 6n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_6$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(6)$:

$$(136) \quad E_{k,6}^0(z) := \frac{6^{k/2}(E_k(6z) - E_k(3z) - E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1}(\Gamma_0(6))\gamma_0 = \Gamma_0(6)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 6n+1 & 3n \\ -12n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{6,3}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(6)$:

$$(137) \quad E_{k,6}^{-1/2}(z) := \frac{-3^{k/2}(2^k E_k(6z) - E_k(3z) - 2^k E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1}(\Gamma_0(6))\gamma_{-1/2} = \Gamma_0(6)$.

For the cusp $-1/3$. We have $\Gamma_{-1/3} = \{\pm \begin{pmatrix} 6n+1 & 2n \\ -18n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/3} = W_{6,2}$, and we have the Eisenstein series for the cusp $-1/3$ associated with $\Gamma_0(6)$:

$$(138) \quad E_{k,6}^{-1/3}(z) := \frac{-2^{k/2}(3^k E_k(6z) - 3^k E_k(3z) - E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/3}^{-1}(\Gamma_0(6))\gamma_{-1/3} = \Gamma_0(6)$.

The space of modular forms. We have $M_k(\Gamma_0(6)) = \mathbb{C}E_{k,6}^\infty \oplus \mathbb{C}E_{k,6}^0 \oplus \mathbb{C}E_{k,6}^{-1/2} \oplus \mathbb{C}E_{k,6}^{-1/3} \oplus S_k(\Gamma_0(6))$ and $S_k(\Gamma_0(6)) = \Delta_6 M_{k-4}(\Gamma_0(6))$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(6)) = E_{2,6+6}' M_{4n}(\Gamma_0(6)) \oplus \mathbb{C}E_{2,6+3}'(\Delta_6)^n \oplus \mathbb{C}E_{2,6+2}'(\Delta_6)^n$ and

$$\begin{aligned} M_{4n}(\Gamma_0(6)) = & \mathbb{C}(E_{4,6}^\infty)^n \oplus \mathbb{C}(E_{4,6}^\infty)^{n-1} \Delta_6 \oplus \dots \oplus \mathbb{C}E_{4,6}^\infty(\Delta_6)^{n-1} \\ & \oplus \mathbb{C}(E_{4,6}^0)^n \oplus \mathbb{C}(E_{4,6}^0)^{n-1} \Delta_6 \oplus \dots \oplus \mathbb{C}E_{4,6}^0(\Delta_6)^{n-1} \\ & \oplus \mathbb{C}(E_{4,6}^{-1/2})^n \oplus \mathbb{C}(E_{4,6}^{-1/2})^{n-1} \Delta_6 \oplus \dots \oplus \mathbb{C}E_{4,6}^{-1/2}(\Delta_6)^{n-1} \\ & \oplus \mathbb{C}(E_{4,6}^{-1/3})^n \oplus \mathbb{C}(E_{4,6}^{-1/3})^{n-1} \Delta_6 \oplus \dots \oplus \mathbb{C}E_{4,6}^{-1/3}(\Delta_6)^{n-1} \oplus \mathbb{C}(\Delta_6)^n. \end{aligned}$$

Here, we have $E_{2,6+6}' = -72(\Delta_6^\infty)^2 + (\Delta_6^0)^2$, $E_{2,6+3}' = 72(\Delta_6^\infty)^2 + 18\Delta_6^\infty \Delta_6^0 + (\Delta_6^0)^2$, $E_{2,6+2}' = 72(\Delta_6^\infty)^2 + 16\Delta_6^\infty \Delta_6^0 + (\Delta_6^0)^2$, $E_{4,6}^\infty = (2592/5)(\Delta_6^\infty)^3 \Delta_6^0 + (972/5)(\Delta_6^\infty)^2 (\Delta_6^0)^2 + (121/5)\Delta_6^\infty (\Delta_6^0)^3 + (\Delta_6^0)^4$, $(5/36)E_{4,6}^0 = 720(\Delta_6^\infty)^4 + 242(\Delta_6^\infty)^3 \Delta_6^0 + 27(\Delta_6^\infty)^2 (\Delta_6^0)^2 + \Delta_6^\infty (\Delta_6^0)^3$, $(-5/9)E_{4,6}^{-1/2} = 32(\Delta_6^\infty)^3 \Delta_6^0 + 12(\Delta_6^\infty)^2 (\Delta_6^0)^2 + \Delta_6^\infty (\Delta_6^0)^3$, and $(-5/4)E_{4,6}^{-1/3} = 162(\Delta_6^\infty)^3 \Delta_6^0 + 27(\Delta_6^\infty)^2 (\Delta_6^0)^2 + \Delta_6^\infty (\Delta_6^0)^3$. Now, we can write

$$M_{2n}(\Gamma_0(6)) = \mathbb{C}(\Delta_6^\infty)^{2n} \oplus \mathbb{C}(\Delta_6^\infty)^{2n-1} \Delta_6^0 \oplus \dots \oplus \mathbb{C}(\Delta_6^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(6)$:

$$(139) \quad J_6 := \Delta_6^0 / \Delta_6^\infty (= \eta^5(z) \eta^{-1}(2z) \eta(3z) \eta^{-5}(6z)) = \frac{1}{q} - 5 + 6q + 4q^2 - 3q^3 - \dots,$$

where $v_\infty(J_6) = -1$ and $v_0(J_6) = 1$. Then, we have

$$(140) \quad J_6 : \partial \mathbb{F}_6 \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-9, 0] \subset \mathbb{R}.$$

7. LEVEL 7

We have $\Gamma_0(7)+ = \Gamma_0^*(7)$ and $\Gamma_0(7)- = \Gamma_0(7)$.

We have $W_7 = \begin{pmatrix} 0 & -1/\sqrt{7} \\ \sqrt{7} & 0 \end{pmatrix}$, and denote $\rho_{7,1} := -1/2 + i/(2\sqrt{7})$, $\rho_{7,2} := -5/14 + \sqrt{3}i/14$, and $\rho_{7,3} := 5/14 + \sqrt{3}i/14$. We define

$$(141) \quad \begin{aligned} \Delta_7^\infty(z) &:= \sqrt{\eta^7(7z)/\eta(z)}, & \Delta_7^0(z) &:= \sqrt{\eta^7(z)/\eta(7z)}, \\ \Delta_7(z) &:= \Delta_7^\infty(z)\Delta_7^0(z) = \eta^3(z)\eta^3(7z), \end{aligned}$$

where Δ_7^∞ and Δ_7^0 are 4th semimodular forms for $\Gamma_0(7)$ of weight $3/2$ such that $v_\infty(\Delta_7^\infty) = v_0(\Delta_7^0) = 1$ and $\Delta_7^\infty(\Delta_7^0)^3$ is a modular form for $\Gamma_0(7)$, and Δ_7 is a cusp form for $\Gamma_0(7)$ and 2nd semimodular form for $\Gamma_0^*(7)$ of weight 6. Furthermore, we define

$$(142) \quad E_{2,7}'(z) := (7E_2(7z) - E_2(z))/6, \quad E_{1,7}' := \sqrt{E_{2,7}'},$$

where $\sqrt{\cdot}$ is selected so that constant term of its Fourier expansion is positive. Here, $E_{1,7}'$ is a 2nd semimodular form for $\Gamma_0(7)$ and 4th semimodular form for $\Gamma_0^*(7)$ such that $v_{\rho_{7,2}}(E_{1,7}') = 1$, and $E_{1,7}'\Delta_7$ is a modular form. Then, $E_{2,7}'$ is a square of it.

7.1. $\Gamma_0^*(7)$. (see [SJ2])

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(7)$ as follows:

$$(143) \quad \begin{aligned} \mathbb{F}_{7+} = & \left\{ |z + 1/2| \geq 1/(2\sqrt{7}), -1/2 \leq \operatorname{Re}(z) < -5/14 \right\} \cup \left\{ |z| \geq 1/\sqrt{7}, -5/14 \leq \operatorname{Re}(z) \leq 0 \right\} \\ & \cup \left\{ |z| > 1/\sqrt{7}, 0 < \operatorname{Re}(z) \leq 5/14 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{7}), 5/14 < \operatorname{Re}(z) < 1/2 \right\}. \end{aligned}$$

where $W_7 : e^{i\theta}/\sqrt{7} \rightarrow e^{i(\pi-\theta)}/\sqrt{7}$ and $\begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix} W_7 : e^{i\theta}/(2\sqrt{7}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{7}) - 1/2$. Then, we have

$$(144) \quad \Gamma_0^*(7) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_7, \begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix} \rangle.$$

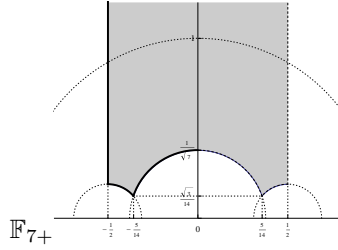


FIGURE 22. $\Gamma_0^*(7)$

Valence formula. The cusp of $\Gamma_0^*(7)$ is ∞ , and the elliptic points are $i/\sqrt{7}$, $\rho_{7,1} = -1/2 + i\sqrt{7}/10$ and $\rho_{7,2} = -5/14 + \sqrt{3}i/14$. Let f be a modular function of weight k for $\Gamma_0^*(7)$, which is not identically zero. We have

$$(145) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{7}}(f) + \frac{1}{2}v_{\rho_{7,1}}(f) + \frac{1}{3}v_{\rho_{7,2}}(f) + \sum_{\substack{p \in \Gamma_0^*(7) \backslash \mathbb{H} \\ p \neq i/\sqrt{7}, \rho_{7,1}, \rho_{7,2}}} v_p(f) = \frac{k}{3}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{7}$ (resp. $\rho_{7,1}$, $\rho_{7,2}$) is $\{\pm I, \pm W_7\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 4 & -1 \\ -7 & 2 \end{pmatrix} W_7\}$, $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix}, \pm \begin{pmatrix} -2 & -1 \\ 7 & 3 \end{pmatrix}\}$)

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0^*(7)$:

$$(146) \quad E_{k,7+}(z) := \frac{7^{k/2}E_k(7z) + E_k(z)}{7^{k/2} + 1} \quad \text{for } k \geq 4.$$

The space of modular forms. We define the following functions:

$$\begin{aligned}\Delta_{4,7} &:= (5/16)((E_{2,7}')^2 - E_{4,7+}), \\ \Delta_{10,0,7+} &:= (559/690)((41065/137592)(E_{4,7+}E_{6,7+} - E_{10,7+}) - E_{6,7+}\Delta_{4,7}),\end{aligned}$$

Here, $\Delta_{4,7}$ and $\Delta_{10,0,7+}$ are cusp forms for $\Gamma_0^*(7)$ and $\Gamma_0(7)$ of weight 4 and 10 such that $v_\infty(\Delta_{4,7}) = v_{\rho_{7,2}}(\Delta_{4,7}) = 1$ and $v_{i/\sqrt{7}}(\Delta_{10,0,7+}) = v_{\rho_{7,1}}(\Delta_{10,0,7+}) = 1$, $v_\infty(\Delta_{10,0,7+}) = 2$, respectively.

Furthermore, we have $M_k(\Gamma_0^*(7)) = \mathbb{C}E_{k,7+} \oplus S_k(\Gamma_0^*(7))$ and $S_k(\Gamma_0^*(7)) = S_{12}(\Gamma_0^*(7))M_{k-12}(\Gamma_0^*(7))$, where

$$S_{12}(\Gamma_0^*(7)) = \mathbb{C}(E_{2,7}')^4\Delta_{7,4} \oplus \mathbb{C}(E_{2,7}')^2(\Delta_{7,4})^2 \oplus \mathbb{C}(\Delta_{7,4})^3 \oplus \mathbb{C}(\Delta_7)^4$$

Then, when we write $k = 12 + l$ for $l = 0, 4, 6, 8, 10, 14$, we have

$$\begin{aligned}M_{12+l}(\Gamma_0^*(7)) &= E_{l,7+}(\mathbb{C}(E_{2,7}')^{6n} \oplus (E_{2,7}')^{6(n-1)}S_{12}(\Gamma_0^*(7)) \oplus (E_{2,7}')^{6(n-2)}(\Delta_7)^4S_{12}(\Gamma_0^*(7)) \\ &\quad \oplus \dots \oplus (\Delta_7)^{4(n-1)}S_{12}(\Gamma_0^*(7))) \\ &\quad \oplus (\Delta_7)^{4n}S_l(\Gamma_0^*(7)).\end{aligned}$$

where we denote $\Delta_{6,7+} := \Delta_{10,0,7+}/\Delta_{4,7}$, and we have $S_4(\Gamma_0^*(7)) = \mathbb{C}\Delta_{7,4}$, $S_6(\Gamma_0^*(7)) = \mathbb{C}\Delta_{6,7+}$,

$$\begin{aligned}S_8(\Gamma_0^*(7)) &= \mathbb{C}(E_{2,7}')^2\Delta_{7,4} \oplus \mathbb{C}(\Delta_{7,4})^2, \quad S_{10}(\Gamma_0^*(7)) = \mathbb{C}(E_{2,7}')^2\Delta_{6,7+} \oplus \mathbb{C}\Delta_{10,0,7+}, \\ S_{14}(\Gamma_0^*(7)) &= \mathbb{C}(E_{2,7}')^4\Delta_{6,7+} \oplus \mathbb{C}(E_{2,7}')^2\Delta_{10,0,7+} \oplus \mathbb{C}\Delta_{7,4}\Delta_{10,0,7+}.\end{aligned}$$

Here, we define

$$E_{3,7}' := \Delta_{10,0,7+}/(\Delta_{4,7}\Delta_7),$$

which is 4th semimodular form such that $v_{i/\sqrt{7}}(E_{3,7}') = v_{\rho_{7,1}}(E_{3,7}') = 1$, and $(E_{1,7}')^3E_{3,7}'$ is a modular form.

Then, we have $\Delta_{4,7} = E_{1,7}'\Delta_7$, $\Delta_{6,7+} = E_{3,7}'\Delta_7$, and $\Delta_{10,0,7+} = E_{1,7}'E_{3,7}'(\Delta_7)^2$, and

$$S_k(\Gamma_0^*(7)) = (\mathbb{C}(E_{1,7}')^9\Delta_7 \oplus \mathbb{C}(E_{1,7}')^6(\Delta_7)^2 \oplus \mathbb{C}(E_{1,7}')^3(\Delta_7)^3 \oplus \mathbb{C}(\Delta_7)^4)M_{k-12}(\Gamma_0^*(7)).$$

Furthermore, we can write

$$M_k(\Gamma_0^*(7)) = E_{\bar{k},7+}'(\mathbb{C}((E_{1,7}')^3)^n \oplus \mathbb{C}((E_{1,7}')^3)^{n-1}\Delta_7 \oplus \dots \oplus \mathbb{C}(\Delta_7)^n),$$

where $n = \dim(M_k(\Gamma_0^*(7))) - 1 = \lfloor k/3 - 2(k/4 - \lfloor k/4 \rfloor) \rfloor$, and where $E_{\bar{k},7+}' := 1$, $(E_{1,7}')^2E_{3,7}'$, $E_{1,7}'$, $E_{3,7}'$, $(E_{1,7}')^2$, and $E_{1,7}'E_{3,7}'$, when $k \equiv 0, 2, 4, 6, 8$, and $10 \pmod{12}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(7)$:

$$(147) \quad J_{7+} := (E_{1,7}')^3/\Delta_7 = \frac{1}{q} + 9 + 51q + 204q^2 + 681q^3 + \dots,$$

where $v_\infty(J_{7+}) = -1$ and $v_{\rho_{7,2}}(J_{7+}) = 3$. Then, we have

$$(148) \quad J_{7+} : \partial\mathbb{F}_{7+} \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [-1, 27] \subset \mathbb{R}.$$

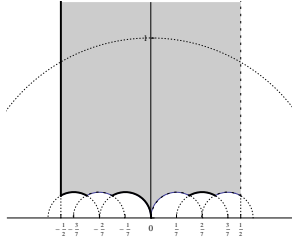
7.2. $\Gamma_0(7)$. (see [SJ1])

Fundamental domain. We have a fundamental domain for $\Gamma_0(7)$ as follows:

$$(149) \quad \begin{aligned}\mathbb{F}_7 &= \{|z + 3/7| \geq 1/7, -1/2 \leq Re(z) \leq -5/14\} \cup \{|z + 2/7| > 1/7, -5/14 < Re(z) < -3/14\} \\ &\cup \{|z + 1/7| \geq 1/7, -3/14 \leq Re(z) \leq 0\} \cup \{|z - 1/7| > 1/7, 0 < Re(z) < 3/14\} \\ &\cup \{|z - 2/7| \geq 1/7, 3/14 \leq Re(z) \leq 5/14\} \cup \{|z - 3/7| > 1/7, 5/14 < Re(z) < 1/2\}.\end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 7 & -1 \end{pmatrix} : (e^{i\theta} + 1)/7 \rightarrow (e^{i(\pi-\theta)} - 1)/7$, $\begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix} : (e^{i\theta} - 2)/7 \rightarrow (e^{i(\pi-\theta)} - 3)/7$, and $\begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix} : (e^{i\theta} + 3)/7 \rightarrow (e^{i(\pi-\theta)} + 2)/7$. Then, we have

$$(150) \quad \Gamma_0(7) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 0 \\ 7 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix} \rangle.$$

FIGURE 23. $\Gamma_0(7)$

Valence formula. The cusps of $\Gamma_0(7)$ are ∞ and 0, and the elliptic points are $\rho_{7,2}$ and $\rho_{7,3} = 5/14 + \sqrt{3}i/14$. Let f be a modular function of weight k for $\Gamma_0(7)$, which is not identically zero. We have

$$(151) \quad v_\infty(f) + v_0(f) + \frac{1}{3}v_{\rho_{7,2}}(f) + \frac{1}{3}v_{\rho_{7,3}}(f) + \sum_{\substack{p \in \Gamma_0(7) \backslash \mathbb{H} \\ p \neq \rho_{7,2}, \rho_{7,3}}} v_p(f) = \frac{2k}{3}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{7,2}$ (resp. $\rho_{7,3}$) is $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 7 & 2 \end{pmatrix}, \pm \begin{pmatrix} -2 & -1 \\ 7 & 3 \end{pmatrix}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 3 & -1 \\ 7 & -3 \end{pmatrix}, \pm \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(7)$:

$$(152) \quad E_{k,7}^\infty(z) := \frac{7^k E_k(7z) - E_k(z)}{7^k - 1} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 7n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_7$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(7)$:

$$(153) \quad E_{k,7}^0(z) := \frac{-7^{k/2}(E_k(7z) - E_k(z))}{7^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(7) \gamma_0 = \Gamma_0(7)$.

The space of modular forms. We define the following functions:

$$\Delta_{6,0,7} := \Delta_7^\infty (\Delta_7^0)^3, \quad \Delta_{6,1,7} := (\Delta_7^\infty)^3 \Delta_7^0,$$

where they are cusp forms of weight 6.

Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0(7)) = \mathbb{C}E_{k,7}^\infty \oplus \mathbb{C}E_{k,7}^0 \oplus S_k(\Gamma_0(7))$ and $S_k(\Gamma_0(7)) = (\mathbb{C}\Delta_{6,0,7} \oplus \mathbb{C}\Delta_{6,1,7} \oplus \mathbb{C}(\Delta_7^2))M_{k-6}(\Gamma_0(7))$. Then, we have $M_{6n+2}(\Gamma_0(7)) = E_{2,7}' M_{6n}(\Gamma_0(7))$ and

$$\begin{aligned} M_{6n}(\Gamma_0(7)) &= \mathbb{C}(E_{6,7}^\infty)^n \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} \Delta_{6,0,7} \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} (\Delta_7^2) \oplus \cdots \oplus \mathbb{C}\Delta_{6,0,7} (\Delta_7^2)^{(n-1)} \\ &\quad \oplus \mathbb{C}(E_{6,7}^0)^n \oplus \mathbb{C}(E_{6,7}^0)^{n-1} \Delta_{6,1,7} \oplus \mathbb{C}(E_{6,7}^0)^{n-1} (\Delta_7^2) \oplus \cdots \oplus \mathbb{C}\Delta_{6,1,7} (\Delta_7^2)^{(n-1)} \oplus \mathbb{C}(\Delta_7^2)^{2n}, \\ M_{6n+4}(\Gamma_0(7)) &= E_{4,7}^\infty (\mathbb{C}(E_{6,7}^\infty)^n \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} \Delta_{6,0,7} \oplus \mathbb{C}(E_{6,7}^\infty)^{n-1} (\Delta_7^2) \oplus \cdots \oplus \mathbb{C}(\Delta_7^2)^{2n}) \\ &\quad \oplus E_{4,7}^0 (\mathbb{C}(E_{6,7}^0)^n \oplus \mathbb{C}(E_{6,7}^0)^{n-1} \Delta_{6,1,7} \oplus \mathbb{C}(E_{6,7}^0)^{n-1} (\Delta_7^2) \oplus \cdots \oplus \mathbb{C}(\Delta_7^2)^{2n}) \oplus \mathbb{C}\Delta_{4,7} (\Delta_7^2)^{2n}. \end{aligned}$$

Furthermore, we can write

$$M_k(\Gamma_0(7)) = E_{k-3n/2,7}' (\mathbb{C}(\Delta_7^\infty)^n \oplus \mathbb{C}(\Delta_7^\infty)^{n-1} \Delta_7^0 \oplus \cdots \oplus \mathbb{C}(\Delta_7^0)^n),$$

where $n = \dim(M_k(\Gamma_0(7))) - 1 = \lfloor 2k/3 - 2(k/3 - \lfloor k/3 \rfloor) \rfloor$ and where $E_{0,7}' := 1$.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(7)$:

$$(154) \quad J_7 := \Delta_7^0 / \Delta_7^\infty = (\eta^4(z) / \eta^4(7z)) = \frac{1}{q} - 4 + 2q + 8q^2 - 5q^3 - \cdots,$$

where $v_\infty(J_7) = -1$ and $v_0(J_7) = 1$. Then, we have

(155)

$$\begin{aligned} J_7 : \quad & \{|z + 1/7| = 1/7, -3/14 \leq \operatorname{Re}(z) \leq 0\} \rightarrow [-2\sqrt{7} + 1/2, 0] \subset \mathbb{R}, \\ & \{|z + 3/7| = 1/7, -1/2 \leq \operatorname{Re}(z) \leq -5/14\} \rightarrow \{-13/2 \leq \operatorname{Re}(z) \leq -2\sqrt{7} + 1/2, 0 \leq \operatorname{Im}(z) \leq 3\sqrt{3}/2\}, \\ & \{|z - 2/7| = 1/7, 3/14 \leq \operatorname{Re}(z) \leq 5/14\} \rightarrow \{-13/2 \leq \operatorname{Re}(z) \leq -2\sqrt{7} + 1/2, -3\sqrt{3}/2 \leq \operatorname{Im}(z) \leq 0\}. \end{aligned}$$

Thus, J_7 does not take real value on some arcs of $\partial\mathbb{F}_7$.

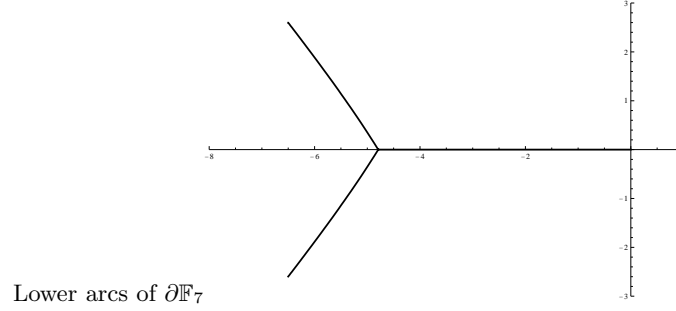


FIGURE 24. Image by J_7

8. LEVEL 8

We have $\Gamma_0(8)+ = \Gamma_0(8) + 8 = \Gamma_0^*(8)$ and $\Gamma_0(8)- = \Gamma_0(8)$.

We have $W_8 = \begin{pmatrix} 0 & -1/(2\sqrt{2}) \\ 2\sqrt{2} & 0 \end{pmatrix}$, $W_{8-,2} := \begin{pmatrix} -1 & -1/2 \\ 4 & 1 \end{pmatrix}$, and $W_{8-,4} := \begin{pmatrix} -\sqrt{2} & -3/(2\sqrt{2}) \\ 2\sqrt{2} & \sqrt{2} \end{pmatrix}$, and we denote $\rho_8 := -1/3 + i/(6\sqrt{2})$. We define

$$(156) \quad \begin{aligned} \Delta_8^\infty(z) &:= \eta^4(8z)/\eta^2(4z), & \Delta_8^0(z) &:= \eta^4(z)/\eta^2(2z), \\ \Delta_8^{-1/2}(z) &:= \eta^{10}(2z)/(\eta^4(z)\eta^4(4z)), & \Delta_8^{-1/4}(z) &:= \eta^{10}(4z)/(\eta^4(2z)\eta^4(8z)), \\ \Delta_8(z) &:= \Delta_8^\infty(z)\Delta_8^0(z)\Delta_8^{-1/2}(z)\Delta_8^{-1/4}(z) = \eta^4(2z)\eta^4(4z), \end{aligned}$$

where Δ_8^∞ , Δ_8^0 , $\Delta_8^{-1/2}$, and $\Delta_8^{-1/4}$ are 2nd semimodular forms for $\Gamma_0(8)$ of weight 1 such that $v_\infty(\Delta_8^\infty) = v_0(\Delta_8^0) = v_{-1/2}(\Delta_8^{-1/2}) = v_{-1/4}(\Delta_8^{-1/4}) = 1$, and Δ_8 is a cusp form for $\Gamma_0(8)$ and $\Gamma_0^*(8)$ of weight 4. Furthermore, we define

$$(157) \quad \begin{aligned} E_{2,8}'(z) &:= 2E_2(8z) - E_2(4z), \\ E_{2,8+8}'(z) &:= (8E_2(8z) - 4E_2(4z) - 2E_2(2z) + E_2(z))/3, \end{aligned}$$

which are modular forms for $\Gamma_0(8)$ of weight 2, and we have $v_{-1/8+i/8}(E_{2,8}') = v_{-3/8+i/8}(E_{2,8}') = 1$ and $v_{i/(2\sqrt{2})}(E_{2,8+8}') = v_{\rho_8}(E_{2,8+8}') = 1$.

8.1. $\Gamma_0(8) + 8 = \Gamma_0^*(8)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(8)$ as follows:

$$(158) \quad \begin{aligned} \mathbb{F}_{8+8} &= \{|z + 3/8| \geq 1/8, -1/2 \leq \operatorname{Re}(z) < -1/4\} \cup \{|z| \geq 1/(2\sqrt{2}), -1/4 \leq \operatorname{Re}(z) \leq 0\} \\ &\cup \{|z| > 1/(2\sqrt{2}), 0 < \operatorname{Re}(z) \leq 1/4\} \cup \{|z - 3/8| > 1/8, 1/4 < \operatorname{Re}(z) < 1/2\}. \end{aligned}$$

where $W_8 : e^{i\theta}/(2\sqrt{2}) \rightarrow e^{i(\pi-\theta)}/(2\sqrt{2})$ and $\begin{pmatrix} -3 & 1 \\ 8 & -3 \end{pmatrix} : (e^{i\theta} + 3)/8 \rightarrow (e^{i(\pi-\theta)} - 3)/8$. Then, we have

$$(159) \quad \Gamma_0^*(8) = \langle (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}), W_8, (\begin{smallmatrix} 3 & 1 \\ 8 & 3 \end{smallmatrix}) \rangle.$$

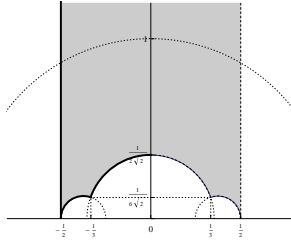


FIGURE 25. $\Gamma_0^*(8)$

Valence formula. The cusps of $\Gamma_0^*(8)$ are ∞ and $-1/2$, and the elliptic points are $i/(2\sqrt{2})$ and $\rho_8 = -1/4 + i/(5\sqrt{6})$. Let f be a modular function of weight k for $\Gamma_0^*(8)$, which is not identically zero. We have

$$(160) \quad v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/(2\sqrt{2})}(f) + \frac{1}{2}v_{\rho_8}(f) + \sum_{\substack{p \in \Gamma_0^*(8) \backslash \mathbb{H} \\ p \neq i/(2\sqrt{2}), \rho_8}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $i/(2\sqrt{2})$ (resp. ρ_8) is $\{\pm I, \pm W_8\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 3 & -1 \\ -8 & 3 \end{pmatrix} W_8\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(8)$:

$$(161) \quad E_{k,8+8}^\infty(z) := \frac{2^{3k/2}E_k(8z) - 2^{k/2}E_k(4z) - E_k(2z) + E_k(z)}{2^{k/2}(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} 4n+1 & 2n \\ -8n & -4n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{8-,4}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0^*(8)$:

$$(162) \quad E_{k,8+8}^{-1/2}(z) := \frac{-(2^{3k/2}E_k(8z) - 2^{k/2}(2^k - 2^{k/2} + 1)E_k(4z) - (2^k - 2^{k/2} + 1)E_k(2z) + E_k(z))}{2^{k/2}(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0^*(8) \gamma_{-1/2} = \Gamma_0^*(8)$.

The space of modular forms. We define

$$\Delta_{8+8}^\infty := \Delta_8^\infty \Delta_8^0, \quad \Delta_{8+8}^{-1/2} := \Delta_8^{-1/2} \Delta_8^{-1/4},$$

which are 2nd semimodular forms for $\Gamma_0^*(8)$ of weight 2.

Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0^*(8)) = \mathbb{C}E_{k,8+8}^\infty \oplus \mathbb{C}E_{k,8+8}^{-1/2} \oplus S_k(\Gamma_0^*(8))$ and $S_k(\Gamma_0^*(8)) = \Delta_8 M_{k-4}(\Gamma_0^*(8))$. Then, we have $M_{4n+2}(\Gamma_0^*(8)) = E_{2,8+8}' M_{4n}(\Gamma_0^*(8))$ and

$$\begin{aligned} M_{4n}(\Gamma_0^*(8)) &= \mathbb{C}(E_{4,8+8}^\infty)^n \oplus \mathbb{C}(E_{4,8+8}^\infty)^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8+8}^\infty (\Delta_8)^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,8+8}^{-1/2})^n \oplus \mathbb{C}(E_{4,8+8}^{-1/2})^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8+8}^{-1/2} (\Delta_8)^{n-1} \oplus \mathbb{C}(\Delta_8)^n. \end{aligned}$$

Furthermore, we can write

$$M_{4n}(\Gamma_0^*(8)) = \mathbb{C}(\Delta_{8+8}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{8+8}^\infty)^{2n-1} \Delta_{8+8}^{-1/2} \oplus \cdots \oplus \mathbb{C}(\Delta_{8+8}^{-1/2})^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(8)$:

$$(163) \quad J_{8+8} := \Delta_{8+8}^{-1/2} / \Delta_{8+8}^\infty (= \eta^{-8}(z) \eta^8(2z) \eta^8(4z) \eta^{-8}(8z)) = \frac{1}{q} + 8 + 36q + 128q^2 + 386q^3 + \cdots,$$

where $v_\infty(J_{8+8}) = -1$ and $v_{-1/2}(J_{8+8}) = 1$. Then, we have

$$(164) \quad J_{8+8} : \partial \mathbb{F}_{8+8} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 12 + 8\sqrt{2}] \subset \mathbb{R}.$$

8.2. $\Gamma_0(8)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(8)$ as follows:

$$(165) \quad \begin{aligned} \mathbb{F}_8 &= \{|z + 3/8| \geq 1/8, -1/2 \leq \operatorname{Re}(z) < -1/4\} \cup \{|z + 1/8| \geq 1/8, -1/4 \leq \operatorname{Re}(z) \leq 0\} \\ &\quad \cup \{|z - 1/8| > 1/8, 0 < \operatorname{Re}(z) \leq 1/4\} \cup \{|z - 3/8| > 1/8, 1/4 < \operatorname{Re}(z) < 1/2\}. \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix} : (e^{i\theta} + 1)/8 \rightarrow (e^{i(\pi-\theta)} - 1)/8$ and $\begin{pmatrix} -3 & 1 \\ 8 & -3 \end{pmatrix} : (e^{i\theta} + 3)/8 \rightarrow (e^{i(\pi-\theta)} - 3)/8$.

$$(166) \quad \Gamma_0(8) = \langle -I, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix} \rangle.$$

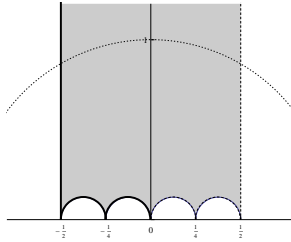


FIGURE 26. $\Gamma_0(8)$

Valence formula. The cusps of $\Gamma_0(8)$ are $\infty, 0, -1/2$, and $-1/4$. Let f be a modular function of weight k for $\Gamma_0(8)$, which is not identically zero. We have

$$(167) \quad v_\infty(f) + v_0(f) + v_{-1/2}(f) + v_{-1/4}(f) + \sum_{p \in \Gamma_0(8) \setminus \mathbb{H}} v_p(f) = k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(8)$:

$$(168) \quad E_{k,8}^\infty(z) := \frac{2^k E_k(8z) - E_k(4z)}{2^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,8}^\infty(z) = E_{k,2}^\infty(4z)$.

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 8n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_8$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(8)$:

$$(169) \quad E_{k,8}^0(z) := \frac{-(E_k(2z) - E_k(z))}{2^{k/2}(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^0(z) = 2^{-k} E_{k,2}^0(z)$. We also have $\gamma_0^{-1} \Gamma_0(8) \gamma_0 = \Gamma_0(8)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 4n+1 & 2n \\ -8n & -4n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{8,-4}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(8)$:

$$(170) \quad E_{k,8}^{-1/2}(z) := \frac{-(2^k E_k(4z) - (2^k + 1)E_k(2z) + E_k(z))}{2^{k/2}(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^{-1/2}(z) = 2^{-k/2} E_{k,4+4}^{-1/2}(z)$. We also have $\gamma_{-1/2}^{-1} \Gamma_0(8) \gamma_{-1/2} = \Gamma_0(8)$.

For the cusp $-1/4$. We have $\Gamma_{-1/4} = \{\pm \begin{pmatrix} 4n+1 & n \\ -16n & -4n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/4} = W_{8,-2}$, and we have the Eisenstein series for the cusp $-1/4$ associated with $\Gamma_0(8)$:

$$(171) \quad E_{k,8}^{-1/4}(z) := \frac{-(2^k E_k(8z) - (2^k + 1)E_k(4z) + E_k(2z))}{2^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,4}^{-1/4}(z) = E_{k,4+4}^{-1/4}(2z)$. We also have $\gamma_{-1/4}^{-1} \Gamma_0(8) \gamma_{-1/4} = \Gamma_0(8)$.

The space of modular forms. Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0(8)) = \mathbb{C}E_{k,8}^\infty \oplus \mathbb{C}E_{k,8}^0 \oplus \mathbb{C}E_{k,8}^{-1/2} \oplus \mathbb{C}E_{k,8}^{-1/4} \oplus S_k(\Gamma_0(8))$ and $S_k(\Gamma_0(8)) = \Delta_8 M_{k-4}(\Gamma_0(8))$. Then, we have $M_{4n+2}(\Gamma_0(8)) = E_{2,8}' M_{4n}(\Gamma_0(8)) \oplus \mathbb{C}E_{2,4}'(\Delta_8)^n \oplus \mathbb{C}E_{2,2}'(\Delta_8)^n$ and

$$\begin{aligned} M_{4n}(\Gamma_0(8)) = & \mathbb{C}(E_{4,8}^\infty)^n \oplus \mathbb{C}(E_{4,8}^\infty)^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8}^\infty(\Delta_8)^{n-1} \\ & \oplus \mathbb{C}(E_{4,8}^0)^n \oplus \mathbb{C}(E_{4,8}^0)^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8}^0(\Delta_8)^{n-1} \\ & \oplus \mathbb{C}(E_{4,8}^{-1/2})^n \oplus \mathbb{C}(E_{4,8}^{-1/2})^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8}^{-1/2}(\Delta_8)^{n-1} \\ & \oplus \mathbb{C}(E_{4,8}^{-1/4})^n \oplus \mathbb{C}(E_{4,8}^{-1/4})^{n-1} \Delta_8 \oplus \cdots \oplus \mathbb{C}E_{4,8}^{-1/4}(\Delta_8)^{n-1} \oplus \mathbb{C}(\Delta_8)^n. \end{aligned}$$

Furthermore, we can write

$$M_{2n}(\Gamma_0(8)) = \mathbb{C}(\Delta_8^\infty)^{2n} \oplus \mathbb{C}(\Delta_8^\infty)^{2n-1} \Delta_8^0 \oplus \cdots \oplus \mathbb{C}(\Delta_8^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(8)$:

$$(172) \quad J_8 := \Delta_8^0 / \Delta_8^\infty (= \eta^4(z) \eta^{-2}(2z) \eta^2(4z) \eta^{-4}(8z)) = \frac{1}{q} - 4 + 4q + 2q^3 - 8q^5 - \cdots,$$

where $v_\infty(J_8) = -1$ and $v_0(J_8) = 1$. Then, we have

$$(173) \quad J_8 : \partial \mathbb{F}_8 \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-8, 0] \subset \mathbb{R}.$$

9. LEVEL 9

We have $\Gamma_0(9)+ = \Gamma_0(9) + 9 = \Gamma_0^*(9)$ and $\Gamma_0(9)- = \Gamma_0(9)$.

We have $W_9 = \begin{pmatrix} 0 & -1/3 \\ 3 & 0 \end{pmatrix}$, $W_{9-,3} := \begin{pmatrix} -1 & -2/3 \\ 3 & 1 \end{pmatrix}$, and $W_{9-, -3} := \begin{pmatrix} 1 & -2/3 \\ 3 & -1 \end{pmatrix}$, and we define $\rho_9 := -1/2 + i/6$. We define

$$(174) \quad \begin{aligned} \Delta_9^\infty(z) &:= \eta^3(9z)/\eta(3z), & \Delta_9^0(z) &:= \eta^3(z)/\eta(3z), \\ \Delta_9^{-1/3}(z) &:= \eta^3(z+1/3)/\eta(3z), & \Delta_9^{1/3}(z) &:= \eta^3(z-1/3)/\eta(3z), \\ \Delta_9(z) &:= \Delta_9^\infty(z)\Delta_9^0(z)\Delta_9^{-1/3}(z)\Delta_9^{1/3}(z) = \eta^3(8z), \end{aligned}$$

where Δ_9^∞ , Δ_9^0 , $\Delta_9^{-1/2}$, and $\Delta_9^{-1/3}$ are 2nd semimodular forms for $\Gamma_0(9)$ of weight 1 such that $v_\infty(\Delta_9^\infty) = v_0(\Delta_9^0) = v_{-1/3}(\Delta_9^{-1/3}) = v_{1/3}(\Delta_9^{1/3}) = 1$, and Δ_9 is a cusp form for $\Gamma_0(9)$ and $\Gamma_0^*(9)$ of weight 4. Furthermore, we define

$$(175) \quad \begin{aligned} E_{2,9'}(z) &:= (3E_2(9z) - E_2(3z))/2, \\ E_{2,9+3'}(z) &:= (3E_2(3z) - E_2(z+1/3))/2, \\ E_{2,9+9'}(z) &:= (9E_2(9z) - 6E_2(3z) + E_2(z))/3, \end{aligned}$$

which are modular forms for $\Gamma_0(9)$ of weight 2, and we have $v_{-1/6+\sqrt{3}i/18}(E_{2,9'}) = 2$, $v_{1/6+\sqrt{3}i/6}(E_{2,9+3'}) = 2$ and $v_{i/3}(E_{2,9+9'}) = v_{\rho_9}(E_{2,9+9'}) = 1$.

9.1. $\Gamma_0(9) + 9 = \Gamma_0^*(9)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(9)$ as follows:

$$(176) \quad \begin{aligned} \mathbb{F}_{9+9} &= \{ |z+5/9| \geq 1/9, -1/2 \leq \operatorname{Re}(z) < -1/3 \} \cup \{ |z| \geq 1/3, -1/3 \leq \operatorname{Re}(z) \leq 0 \} \\ &\cup \{ |z| > 1/3, 0 < \operatorname{Re}(z) \leq 1/3 \} \cup \{ |z-5/9| > 1/9, 1/3 < \operatorname{Re}(z) < 1/2 \}. \end{aligned}$$

where $W_9 : e^{i\theta}/3 \rightarrow e^{i(\pi-\theta)}/3$ and $\begin{pmatrix} -4 & -1 \\ 9 & 2 \end{pmatrix} W_9 : (e^{i\theta}+3)/6 \rightarrow (e^{i(\pi-\theta)}-3)/6$. Then, we have

$$(177) \quad \Gamma_0^*(9) = \langle (\frac{1}{0} \frac{1}{1}), W_9, (\frac{5}{9} \frac{1}{2}) \rangle.$$

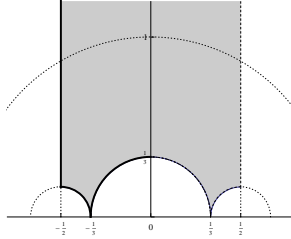


FIGURE 27. $\Gamma_0^*(9)$

Valence formula. The cusps of $\Gamma_0^*(9)$ are ∞ and $-1/3$, and the elliptic points are $i/3$ and $\rho_9 = -1/2 + i/6$. Let f be a modular function of weight k for $\Gamma_0^*(9)$, which is not identically zero. We have

$$(178) \quad v_\infty(f) + v_{-1/3}(f) + \frac{1}{2}v_{i/3}(f) + \frac{1}{2}v_{\rho_9}(f) + \sum_{\substack{p \in \Gamma_0^*(9) \backslash \mathbb{H} \\ p \neq i/3, \rho_9}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $i/3$ (resp. ρ_9) is $\{\pm I, \pm W_9\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 5 & -1 \\ -9 & 2 \end{pmatrix} W_9\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(9)$:

$$(179) \quad E_{k,9+9}^\infty(z) := \frac{3^k E_k(9z) - 2E_k(3z) + E_k(z)}{3^k - 1} \quad \text{for } k \geq 4.$$

For the cusp $-1/3$. We have $\Gamma_{-1/3} = \left\{ \pm \begin{pmatrix} 3n+1 & n \\ -9n & -3n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/3} = W_{9-,3}$, and we have the Eisenstein series for the cusp $-1/3$ associated with $\Gamma_0^*(9)$:

$$(180) \quad E_{k,9+9}^{-1/3}(z) := \frac{-(3^k E_k(9z) - (3^k + 1)E_k(3z) + E_k(z))}{(3^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/3}^{-1} \Gamma_0^*(9) \gamma_{-1/3} = \Gamma_0^*(9)$.

The space of modular forms. We define

$$\Delta_{9+9}^\infty := \Delta_9^\infty \Delta_9^0, \quad \Delta_{9+9}^{-1/3} := \Delta_9^{-1/3} \Delta_9^{1/3},$$

which are 2nd semimodular forms for $\Gamma_0^*(9)$ of weight 2.

Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0^*(9)) = \mathbb{C}E_{k,9+9}^\infty \oplus \mathbb{C}E_{k,9+9}^{-1/3} \oplus S_k(\Gamma_0^*(9))$ and $S_k(\Gamma_0^*(9)) = \Delta_9 M_{k-4}(\Gamma_0^*(9))$. Then, we have $M_{4n+2}(\Gamma_0^*(9)) = E_{2,9+9}' M_{4n}(\Gamma_0^*(9))$ and

$$\begin{aligned} M_{4n}(\Gamma_0^*(9)) &= \mathbb{C}(E_{4,9+9}^\infty)^n \oplus \mathbb{C}(E_{4,9+9}^\infty)^{n-1} \Delta_9 \oplus \cdots \oplus \mathbb{C}E_{4,9+9}^\infty \Delta_9^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,9+9}^{-1/3})^n \oplus \mathbb{C}(E_{4,9+9}^{-1/3})^{n-1} \Delta_9 \oplus \cdots \oplus \mathbb{C}E_{4,9+9}^{-1/3} \Delta_9^{n-1} \oplus \mathbb{C}(\Delta_9)^n. \end{aligned}$$

Furthermore, we can write

$$M_{4n}(\Gamma_0^*(9)) = \mathbb{C}(\Delta_{9+9}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{9+9}^\infty)^{2n-1} \Delta_{9+9}^{-1/3} \oplus \cdots \oplus \mathbb{C}(\Delta_{9+9}^{-1/3})^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(9)$:

$$(181) \quad J_{9+9} := \Delta_{9+9}^{-1/3} / \Delta_{9+9}^\infty (= \eta^{12}(3z) / (\eta^6(z)\eta^6(9z))) = \frac{1}{q} + 6 + 27q + 86q^2 + 243q^3 + \cdots,$$

where $v_\infty(J_{9+9}) = -1$ and $v_{-1/3}(J_{9+9}) = 1$. Then, we have

$$(182) \quad J_{9+9} : \partial\mathbb{F}_{9+9} \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [9 - 6\sqrt{3}, 9 + 6\sqrt{3}] \subset \mathbb{R}.$$

9.2. $\Gamma_0(9)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(9)$ as follows:

$$(183) \quad \begin{aligned} \mathbb{F}_9 &= \{|z + 4/9| \geq 1/9, -1/2 \leq Re(z) \leq -1/3\} \cup \{|z + 2/9| > 1/9, -1/3 < Re(z) < -1/6\} \\ &\cup \{|z + 1/9| \geq 1/9, -1/6 \leq Re(z) \leq 0\} \cup \{|z - 1/9| > 1/9, 0 < Re(z) < 1/6\} \\ &\cup \{|z - 2/9| \geq 1/9, 1/6 \leq Re(z) \leq 1/3\} \cup \{|z - 4/9| > 1/9, 1/3 < Re(z) < 1/2\}. \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 9 & -1 \end{pmatrix} : (e^{i\theta} + 1)/9 \rightarrow (e^{i(\pi-\theta)} - 1)/9$, $\begin{pmatrix} -4 & -1 \\ 9 & 2 \end{pmatrix} : (e^{i\theta} - 2)/9 \rightarrow (e^{i(\pi-\theta)} - 4)/9$, and $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix} : (e^{i\theta} + 4)/9 \rightarrow (e^{i(\pi-\theta)} + 2)/9$.

$$(184) \quad \Gamma_0(9) = \langle -I, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 9 & 2 \end{pmatrix} \rangle.$$

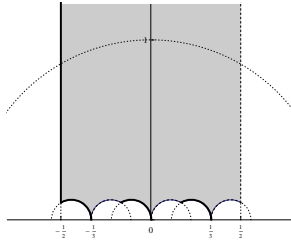


FIGURE 28. $\Gamma_0(9)$

Valence formula. The cusps of $\Gamma_0(9)$ are ∞ , 0 , $-1/3$, and $1/3$. Let f be a modular function of weight k for $\Gamma_0(9)$, which is not identically zero. We have

$$(185) \quad v_\infty(f) + v_0(f) + v_{-1/3}(f) + v_{1/3}(f) + \sum_{p \in \Gamma_0(9) \setminus \mathbb{H}} v_p(f) = k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(9)$:

$$(186) \quad E_{k,9}^\infty(z) := \frac{3^k E_k(9z) - E_k(3z)}{3^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,9}^\infty(z) = E_{k,3}^\infty(3z)$.

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 9n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_9$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(9)$:

$$(187) \quad E_{k,9}^0(z) := \frac{-(E_k(3z) - E_k(z))}{3^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,9}^0(z) = 3^{-k/2} E_{k,3}^0(z)$. We also have $\gamma_0^{-1} \Gamma_0(9) \gamma_0 = \Gamma_0(9)$.

For the cusp $-1/3$. We have $\Gamma_{-1/3} = \{\pm \begin{pmatrix} 3n+1 & 2n \\ -9n & -3n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/3} = W_{9,-3}$, and we have the Eisenstein series for the cusp $-1/3$ associated with $\Gamma_0(9)$:

$$(188) \quad E_{k,9}^{-1/3}(z) := \frac{-(E_k(3z) - E_k(z + 1/3))}{3^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,9}^{-1/3}(z) = 3^{-k/2} E_{k,3}^0(z + 1/3)$. We also have $\gamma_{-1/3}^{-1} \Gamma_0(9) \gamma_{-1/3} = \Gamma_0(9)$.

For the cusp $1/3$. We have $\Gamma_{1/3} = \{\pm \begin{pmatrix} -3n+1 & 2n \\ -9n & 3n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{1/3} = W_{9,-,-3}$, and we have the Eisenstein series for the cusp $1/3$ associated with $\Gamma_0(9)$:

$$(189) \quad E_{k,9}^{1/3}(z) := \frac{-(E_k(3z) - E_k(z - 1/3))}{3^k - 1} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,9}^{1/3}(z) = 3^{-k/2} E_{k,3}^0(z - 1/3)$. We also have $\gamma_{1/3}^{-1} \Gamma_0(9) \gamma_{1/3} = \Gamma_0(9)$.

The space of modular forms. Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0(9)) = \mathbb{C}E_{k,9}^\infty \oplus \mathbb{C}E_{k,9}^0 \oplus \mathbb{C}E_{k,9}^{-1/3} \oplus \mathbb{C}E_{k,9}^{1/3} \oplus S_k(\Gamma_0(9))$ and $S_k(\Gamma_0(9)) = \Delta_9 M_{k-4}(\Gamma_0(9))$. Then, we have $M_{4n+2}(\Gamma_0(9)) = E_{2,9}' M_{4n}(\Gamma_0(9)) \oplus \mathbb{C}E_{2,3}'(\Delta_9)^n \oplus \mathbb{C}E_{2,9+3}'(\Delta_9)^n$ and

$$\begin{aligned} M_{4n}(\Gamma_0(9)) = & \mathbb{C}(E_{4,9}^\infty)^n \oplus \mathbb{C}(E_{4,9}^\infty)^{n-1} \Delta_9 \oplus \dots \oplus \mathbb{C}E_{4,9}^\infty(\Delta_9)^{n-1} \\ & \oplus \mathbb{C}(E_{4,9}^0)^n \oplus \mathbb{C}(E_{4,9}^0)^{n-1} \Delta_9 \oplus \dots \oplus \mathbb{C}E_{4,9}^0(\Delta_9)^{n-1} \\ & \oplus \mathbb{C}(E_{4,9}^{-1/3})^n \oplus \mathbb{C}(E_{4,9}^{-1/3})^{n-1} \Delta_9 \oplus \dots \oplus \mathbb{C}E_{4,9}^{-1/3}(\Delta_9)^{n-1} \\ & \oplus \mathbb{C}(E_{4,9}^{1/3})^n \oplus \mathbb{C}(E_{4,9}^{1/3})^{n-1} \Delta_9 \oplus \dots \oplus \mathbb{C}E_{4,9}^{1/3}(\Delta_9)^{n-1} \oplus \mathbb{C}(\Delta_9)^n, \end{aligned}$$

Furthermore, we can write

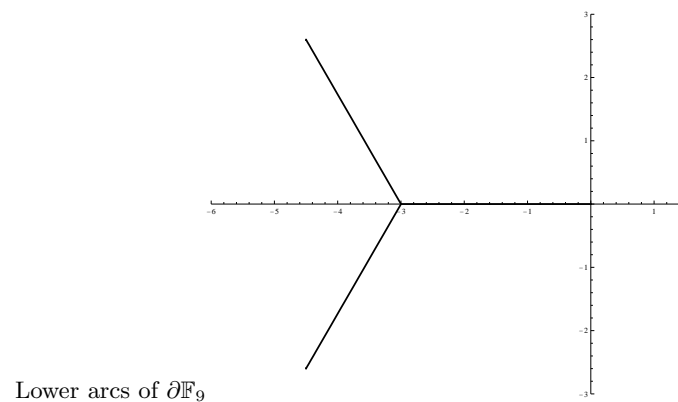
$$M_{2n}(\Gamma_0(9)) = \mathbb{C}(\Delta_9^\infty)^{2n} \oplus \mathbb{C}(\Delta_9^\infty)^{2n-1} \Delta_9^0 \oplus \dots \oplus \mathbb{C}(\Delta_9^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(9)$:

$$(190) \quad J_9 := \Delta_9^0 / \Delta_9^\infty (= \eta^3(z) / \eta^3(9z)) = \frac{1}{q} - 3 + 5q^2 - 7q^5 + 3q^8 + \dots,$$

where $v_\infty(J_9) = -1$ and $v_0(J_9) = 1$. Then, we have

$$(191) \quad \begin{aligned} J_9 : \quad & \{|z + 1/9| = 1/9, -1/6 \leq \operatorname{Re}(z) \leq 0\} \rightarrow -3 + [0, 3] \subset \mathbb{R}, \\ & \{|z + 4/9| = 1/9, -1/2 \leq \operatorname{Re}(z) \leq -1/3\} \rightarrow -3 + e^{2\pi i/3}[0, 3], \\ & \{|z - 2/9| = 1/9, 1/6 \leq \operatorname{Re}(z) \leq 1/3\} \rightarrow -3 + e^{-2\pi i/3}[0, 3]. \end{aligned}$$

FIGURE 29. Image by J_9

10. LEVEL 10

We have $\Gamma_0(10)+$, $\Gamma_0(10) + 10 = \Gamma_0^*(10)$, $\Gamma_0(10) + 5$, $\Gamma_0(10) + 2$, and $\Gamma_0(10)- = \Gamma_0(10)$.

We have $W_{10} = \begin{pmatrix} 0 & -1/\sqrt{10} \\ \sqrt{10} & 0 \end{pmatrix}$, $W_{10,2} := \begin{pmatrix} -\sqrt{2} & -1/\sqrt{2} \\ 5\sqrt{2} & 2\sqrt{2} \end{pmatrix}$, and $W_{10,5} := \begin{pmatrix} -\sqrt{5} & -3/\sqrt{5} \\ 2\sqrt{5} & \sqrt{5} \end{pmatrix}$, and we denote $\rho_{10,1} := -3/10 + i/10$, $\rho_{10,2} := -1/2 + i/(2\sqrt{5})$, $\rho_{10,3} := -1/7 + i/(7\sqrt{10})$, $\rho_{10,4} := -1/6 + i/(6\sqrt{5})$, and $\rho_{10,5} := 3/10 + i/10$. We define

$$(192) \quad \begin{aligned} \Delta_{10}^\infty(z) &:= \sqrt[3]{\eta(z)\eta^{-2}(2z)\eta^{-5}(5z)\eta^{10}(10z)}, & \Delta_{10}^0(z) &:= \sqrt[3]{\eta^{10}(z)\eta^{-5}(2z)\eta^{-2}(5z)\eta(10z)}, \\ \Delta_{10}^{-1/2}(z) &:= \sqrt[3]{\eta^{-5}(z)\eta^{10}(2z)\eta(5z)\eta^{-2}(10z)}, & \Delta_{10}^{-1/5}(z) &:= \sqrt[3]{\eta^{-2}(z)\eta(2z)\eta^{10}(5z)\eta^{-5}(10z)}, \\ \Delta_{10}(z) &:= \Delta_{10}^\infty(z)\Delta_{10}^0(z)\Delta_{10}^{-1/5}(z)\Delta_{10}^{-1/2}(z) = \sqrt[3]{\eta^4(z)\eta^4(2z)\eta^4(5z)\eta^4(10z)}, \end{aligned}$$

where Δ_{10}^∞ , Δ_{10}^0 , $\Delta_{10}^{-1/2}$, and $\Delta_{10}^{-1/5}$ are 6th semimodular forms for $\Gamma_0(10)$ of weight $2/3$ such that $v_\infty(\Delta_{10}^\infty) = v_0(\Delta_{10}^0) = v_{-1/2}(\Delta_{10}^{-1/2}) = v_{-1/5}(\Delta_{10}^{-1/5}) = 1$, and Δ_{10} is a 3rd semimodular form for $\Gamma_0(10)$ of weight $8/3$. Furthermore, we define

$$(193) \quad \begin{aligned} E_{2,10+10}'(z) &:= (10E_2(10z) - 5E_2(5z) - 2E_2(2z) + E_2(z))/4, \\ E_{2,10+5}'(z) &:= (10E_2(10z) - 5E_2(5z) + 2E_2(2z) - E_2(z))/6, \\ E_{2,10+2}'(z) &:= (10E_2(10z) + 5E_2(5z) - 2E_2(2z) - E_2(z))/12, \end{aligned}$$

which are modular forms for $\Gamma_0(10)$ of weight 2, and we have $v_{i/\sqrt{10}}(E_{2,10+10}') = v_{\rho_{10,1}}(E_{2,10+10}') = v_{\rho_{10,3}}(E_{2,10+10}') = 1$, $v_{\rho_{10,1}}(E_{2,10+5}') = v_{\rho_{10,2}}(E_{2,10+5}') = v_{\rho_{10,4}}(E_{2,10+5}') = 1$, and $v_{\rho_{10,1}}(E_{2,10+2}') = v_{\rho_{10,5}}(E_{2,10+2}') = 3$. In addition, we define

$$E_{2/3,10}' := \sqrt[3]{E_{2,10+2}'}, \quad E_{8/3,10}' := E_{2,10+10}'E_{2,10+5}'/(E_{2/3,10}')^2,$$

which are 3rd semimodular forms for $\Gamma_0(10)$ of weight $2/3$ and $8/3$, respectively. We also have $v_{\rho_{10,1}}(E_{2/3,10}') = v_{\rho_{10,5}}(E_{2/3,10}') = 3$ and $v_{i/\sqrt{10}}(E_{8/3,10}') = v_{\rho_{10,2}}(E_{8/3,10}') = v_{\rho_{10,3}}(E_{8/3,10}') = v_{\rho_{10,4}}(E_{8/3,10}') = 1$.

Furthermore, $\prod_{i=1}^6 \Delta_{10}^{\kappa_i}$, $E_{2/3,10}' \Delta_{10}^{\kappa_1} \Delta_{10}^{\kappa_2}$, and $E_{2/3,10}'(E_{8/3,10}')^2$ are modular form for $\Gamma_0(10)$ for $\kappa_i \in \{\infty, 0, -1/2, -1/5\}$.

10.1. $\Gamma_0(10)+$.

We have

$$\Gamma_0(10)+ = \Gamma_0(10) + 2, 5, 10 = \Gamma_0(10) \cup \Gamma_0(10)W_{10,2} \cup \Gamma_0(10)W_{10,5} \cup \Gamma_0(10)W_{10}.$$

Fundamental domain. We have a fundamental domain for $\Gamma_0(10)+$ as follows:

$$(194) \quad \mathbb{F}_{10+} = \left\{ |z + 1/2| \geq 1/(2\sqrt{5}), -1/2 \leq \operatorname{Re}(z) < -3/10 \right\} \cup \left\{ |z| \geq 1/\sqrt{10}, -3/10 \leq \operatorname{Re}(z) \leq 0 \right\} \\ \cup \left\{ |z| > 1/\sqrt{10}, 0 < \operatorname{Re}(z) \leq 3/10 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{5}), 3/10 < \operatorname{Re}(z) < 1/2 \right\},$$

where, $W_{10} : e^{i\theta}/\sqrt{10} \rightarrow e^{i(\pi-\theta)}/\sqrt{10}$ and $\begin{pmatrix} -9 & -5 \\ 20 & 11 \end{pmatrix} W_{10,5} : e^{i\theta}/(2\sqrt{5}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{5}) - 1/2$. Then, we have

$$(195) \quad \Gamma_0(10)+ = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{10}, W_{10,5} \rangle.$$

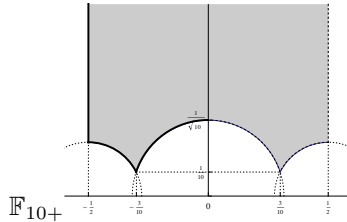


FIGURE 30. $\Gamma_0(10)+$

Valence formula. The cusp of $\Gamma_0(10)+$ is ∞ , and the elliptic points are $i/\sqrt{10}$, $\rho_{10,1} = -3/10 + i/10$, and $\rho_{10,2} = -1/2 + i/(2\sqrt{5})$. Let f be a modular function of weight k for $\Gamma_0(10)+$, which is not identically zero. We have

$$(196) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{10}}(f) + \frac{1}{4}v_{\rho_{10,1}}(f) + \frac{1}{2}v_{\rho_{10,2}}(f) + \sum_{\substack{p \in \Gamma_0(10)+ \setminus \mathbb{H} \\ p \neq i/\sqrt{10}, \rho_{10,1}, \rho_{10,2}}} v_p(f) = \frac{3k}{8}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{10}$ (resp. $\rho_{10,1}$, $\rho_{10,2}$) is $\{\pm I, \pm W_{10}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}, \pm W_{10,2}, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} W_{10,2}\}$, $\{\pm I, \pm W_{10,5}\}$)

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0(10)+$:

$$(197) \quad E_{k,10+}(z) := \frac{10^{k/2}E_k(10z) + 5^{k/2}E_k(5z) + 2^{k/2}E_k(2z) + E_k(z)}{(5^{k/2} + 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

The space of modular forms. Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0(10)+) = \mathbb{C}E_{k,10+} \oplus S_k(\Gamma_0(10)+)$ and $S_k(\Gamma_0(10)+) = (\mathbb{C}(E_{2/3,10}')^8 \Delta_{10} \oplus \mathbb{C}(E_{2/3,10}')^4 (\Delta_{10})^2 \oplus \mathbb{C}(\Delta_{10})^3) M_{k-8}(\Gamma_0(10)+)$. Then, we have

$$M_k(\Gamma_0(10)+) = E_{k,10+}'(\mathbb{C}((E_{2/3,10}')^4)^n \oplus \mathbb{C}((E_{2/3,10}')^4)^{n-1} \Delta_{10} \oplus \cdots \oplus \mathbb{C}(\Delta_{10})^n),$$

where $n = \dim(M_k(\Gamma_0(10)+)) - 1 = \lfloor 3k/8 - 2(k/4 - \lfloor k/4 \rfloor) \rfloor$, and where $E_{k,10+}' := 1, (E_{2/3,10}')^3 E_{8/3,10}', (E_{2/3,10}')^2$, and $E_{2/3,10}' E_{8/3,10}'$, when $k \equiv 0, 2, 4$, and $6 \pmod{8}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(10)+$:

$$(198) \quad J_{10+} := (E_{2/3,10}')^4 / \Delta_{10} = \frac{1}{q} + 4 + 22q + 56q^2 + 177q^3 + \cdots,$$

where $v_\infty(J_{10+}) = -1$ and $v_{\rho_{10,1}}(J_{10+}) = 4$. Then, we have

$$(199) \quad J_{10+} : \partial \mathbb{F}_{10+} \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [-4, 16] \subset \mathbb{R}.$$

10.2. $\Gamma_0(10) + 10 = \Gamma_0^*(10)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(10)$ as follows:

$$(200) \quad \begin{aligned} \mathbb{F}_{10+10} = & \{ |z + 9/20| \geq 1/20, -1/2 \leq Re(z) < -3/7 \} \cup \{ |z + 1/3| \geq 1/(3\sqrt{10}), -3/7 \leq Re(z) < -3/10 \} \\ & \cup \{ |z| \geq 1/\sqrt{10}, -3/10 \leq Re(z) \leq 0 \} \cup \{ |z| > 1/\sqrt{10}, 0 < Re(z) \leq 3/10 \} \\ & \cup \{ |z - 1/3| > 1/(3\sqrt{10}), 3/10 < Re(z) \leq 3/7 \} \cup \{ |z - 9/20| > 1/20, 3/7 < Re(z) < 1/2 \}, \end{aligned}$$

where $W_{10} : e^{i\theta}/\sqrt{10} \rightarrow e^{i(\pi-\theta)}/\sqrt{10}$, $\begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} W_{10} : e^{i\theta}/(3\sqrt{10}) + 1/3 \rightarrow e^{i(\pi-\theta)}/(3\sqrt{10}) - 1/3$, and $\begin{pmatrix} -9 & 4 \\ 20 & -9 \end{pmatrix} : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(201) \quad \Gamma_0^*(10) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{10}, \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}, \begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix} \rangle.$$

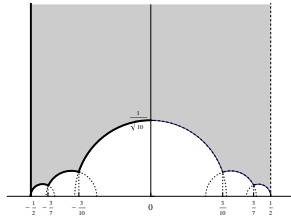


FIGURE 31. $\Gamma_0^*(10)$

Valence formula. The cusps of $\Gamma_0^*(10)$ are ∞ and $-1/2$, and the elliptic points are $i/\sqrt{10}$ and $\rho_{10,5} = -3/7 + i/(5\sqrt{10})$. Let f be a modular function of weight k for $\Gamma_0^*(10)$, which is not identically zero. We have

$$(202) \quad v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/\sqrt{10}}(f) + \frac{1}{2}v_{\rho_{10,1}}(f) + \frac{1}{2}v_{\rho_{10,3}}(f) + \sum_{\substack{p \in \Gamma_0^*(10) \setminus \mathbb{H} \\ p \neq i/\sqrt{10}, \rho_{10,1}, \rho_{10,3}}} v_p(f) = \frac{3k}{4}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{10}$ (resp. $\rho_{10,1}$, $\rho_{10,3}$) is $\{\pm I, \pm W_{10}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}\}$, $\{\pm I, \pm \begin{pmatrix} -13 & 3 \\ 30 & -7 \end{pmatrix} W_{10}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(10)$:

$$(203) \quad E_{k,10+10}^\infty(z) := \frac{(10^{k/2} + 1)(10^{k/2}E_k(10z) + E_k(z)) - (5^{k/2} + 2^{k/2})(5^{k/2}E_k(5z) + 2^{k/2}E_k(2z))}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 10n+1 & 5n \\ -20n & -10n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{10,5}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0^*(10)$:

$$(204) \quad E_{k,10+10}^{-1/2}(z) := \frac{-(5^{k/2} + 2^{k/2})(10^{k/2}E_k(10z) + E_k(z)) + (10^{k/2} + 1)(5^{k/2}E_k(5z) + 2^{k/2}E_k(2z))}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0^*(10) \gamma_{-1/2} = \Gamma_0^*(10)$.

The space of modular forms. We define

$$\Delta_{10+10}^\infty := \Delta_{10}^\infty \Delta_{10}^0, \quad \Delta_{10+10}^{-1/2} := \Delta_{10}^{-1/2} \Delta_{10}^{-1/5},$$

which are 2nd semimodular forms for $\Gamma_0^*(10)$ of weight 2. Furthermore, we define

$$\begin{aligned} \Delta_{8,1,10+10} &:= \Delta_{10+10}^\infty (\Delta_{10+10}^{-1/2})^5, & \Delta_{8,2,10+10} &:= (\Delta_{10+10}^\infty)^5 \Delta_{10+10}^{-1/2}, \\ \Delta_{8,3,10+10} &:= (\Delta_{10+10}^\infty)^2 (\Delta_{10+10}^{-1/2})^4, & \Delta_{8,4,10+10} &:= (\Delta_{10+10}^\infty)^4 (\Delta_{10+10}^{-1/2})^2. \end{aligned}$$

Now, we have

$$M_k(\Gamma_0^*(10)) = \mathbb{C}E_{k,10+10}^\infty \oplus \mathbb{C}E_{k,10+10}^{-1/2} \oplus S_k(\Gamma_0^*(10)),$$

$$S_k(\Gamma_0^*(10)) = (\mathbb{C}\Delta_{8,1,10+10} \oplus \mathbb{C}\Delta_{8,2,10+10} \oplus \mathbb{C}\Delta_{8,3,10+10} \oplus \mathbb{C}\Delta_{8,4,10+10} \oplus \mathbb{C}(\Delta_{10})^3)M_{k-8}(\Gamma_0^*(10))$$

for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0^*(10)) = E_{2,10+10}' M_{4n}(\Gamma_0^*(10))$ and

$$\begin{aligned} M_{8n}(\Gamma_0^*(10)) &= \mathbb{C}(E_{8,10+10}^\infty)^n \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} \Delta_{8,1,10+10} \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} \Delta_{8,3,10+10} \\ &\quad \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,3,10+10} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^n \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} \Delta_{8,2,10+10} \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} \Delta_{8,4,10+10} \\ &\quad \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,4,10+10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}, \\ M_{8n+4}(\Gamma_0^*(10)) &= E_{4,10+10}^\infty (\mathbb{C}(E_{8,10+10}^\infty)^n \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} \Delta_{8,1,10+10} \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} \Delta_{8,3,10+10} \\ &\quad \oplus \mathbb{C}(E_{8,10+10}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,3,10+10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus E_{4,10+10}^{-1/2} (\mathbb{C}(E_{8,10+10}^{-1/2})^n \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} \Delta_{8,2,10+10} \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} \Delta_{8,4,10+10} \\ &\quad \oplus \mathbb{C}(E_{8,10+10}^{-1/2})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,4,10+10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus (\mathbb{C}\Delta_{10+10}^\infty (\Delta_{10+10}^{-1/2})^2 \oplus \mathbb{C}(\Delta_{10+10}^\infty)^2 \Delta_{10+10}^{-1/2}) (\Delta_{10})^{3n}. \end{aligned}$$

Furthermore, we can write

$$M_{4n}(\Gamma_0^*(10)) = \mathbb{C}(\Delta_{10+10}^\infty)^{3n} \oplus \mathbb{C}(\Delta_{10+10}^\infty)^{3n-1} \Delta_{10+10}^{-1/2} \oplus \cdots \oplus \mathbb{C}(\Delta_{10+10}^{-1/2})^{3n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(10)$:

$$(205) \quad J_{10+10} := \Delta_{10+10}^{-1/2} / \Delta_{10+10}^\infty (= \eta^{-6}(z)\eta^6(2z)\eta^{-6}(5z)\eta^6(10z)) = \frac{1}{q} + 6 + 21q + 62q^2 + 162q^3 + \cdots,$$

where $v_\infty(J_{10+10}) = -1$ and $v_{-1/2}(J_{10+10}) = 1$. Then, we have

$$(206) \quad J_{10+10} : \partial\mathbb{F}_{10+10} \setminus \{z \in \mathbb{H}; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 9 + 4\sqrt{5}] \subset \mathbb{R}.$$

10.3. $\Gamma_0(10) + 5$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(10) + 5$ as follows:

$$(207) \quad \mathbb{F}_{10+5} = \left\{ |z + 1/2| \geq 1/(2\sqrt{5}), -1/2 \leq \operatorname{Re}(z) < -3/10 \right\} \cup \left\{ |z + 1/4| \geq 1/(4\sqrt{5}), -3/10 \leq \operatorname{Re}(z) < -1/6 \right\} \\ \cup \left\{ |z + 1/10| \geq 1/10, -1/6 \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z - 1/10| > 1/10, 0 < \operatorname{Re}(z) \leq 1/6 \right\} \\ \cup \left\{ |z - 1/4| > 1/(4\sqrt{5}), 1/6 < \operatorname{Re}(z) \leq 3/10 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{5}), 3/10 < \operatorname{Re}(z) < 1/2 \right\},$$

where $\begin{pmatrix} -1 & 0 \\ 10 & -1 \end{pmatrix} : (e^{i\theta} + 1)/10 \rightarrow (e^{i(\pi-\theta)} - 1)/10$, $\begin{pmatrix} -7 & -4 \\ 30 & 17 \end{pmatrix} W_{10,5} : e^{i\theta}/(4\sqrt{5}) + 1/4 \rightarrow e^{i(\pi-\theta)}/(4\sqrt{5}) - 1/4$, and $\begin{pmatrix} -9 & -5 \\ 20 & 11 \end{pmatrix} W_{10,5} : e^{i\theta}/(2\sqrt{5}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{5}) - 1/2$. Then, we have

$$(208) \quad \Gamma_0(10) + 5 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{10,5}, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} \rangle.$$

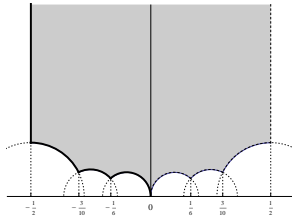


FIGURE 32. $\Gamma_0(10) + 5$

Valence formula. The cusps of $\Gamma_0(10) + 5$ are ∞ and 0 , and the elliptic points are $\rho_{10,1} = -3/10 + i/10$, $\rho_{10,2} = -1/2 + i/(2\sqrt{5})$, and $\rho_{10,4} = -1/6 + i/(6\sqrt{5})$. Let f be a modular function of weight k for $\Gamma_0(10) + 5$, which is not identically zero. We have

$$(209) \quad v_\infty(f) + v_0(f) + \frac{1}{2}v_{\rho_{10,1}}(f) + \frac{1}{2}v_{\rho_{10,2}}(f) + \frac{1}{2}v_{\rho_{10,4}}(f) + \sum_{\substack{p \in \Gamma_0(10)+5 \backslash \mathbb{H} \\ p \neq \rho_{10,1}, \rho_{10,2}, \rho_{10,4}}} v_p(f) = \frac{3k}{4}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{10,1}$ (resp. $\rho_{10,2}, \rho_{10,4}$) is $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}\}$ (resp. $\{\pm I, \pm W_{10,5}\}, \{\pm I, \pm \begin{pmatrix} -3 & -2 \\ 20 & 13 \end{pmatrix} W_{10,5}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(10) + 5$:

$$(210) \quad E_{k,10+5}^\infty(z) := \frac{2^k 5^{k/2} E_k(10z) - 5^{k/2} E_k(5z) + 2^k E_k(2z) - E_k(z)}{(5^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp 0 . We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 10n & 1 \end{pmatrix}; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{10}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(10) + 5$:

$$(211) \quad E_{k,10+5}^0(z) := \frac{-2^{k/2} (5^{k/2} E_k(10z) - 5^{k/2} E_k(5z) + E_k(2z) - E_k(z))}{(5^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} (\Gamma_0(10) + 5) \gamma_0 = \Gamma_0(10) + 5$.

The space of modular forms. We define

$$\Delta_{10+5}^\infty := \Delta_{10}^\infty \Delta_{10}^{-1/2}, \quad \Delta_{10+5}^0 := \Delta_{10}^0 \Delta_{10}^{-1/5},$$

which are 2nd semimodular forms for $\Gamma_0(10) + 5$ of weight 2. Furthermore, we define

$$\begin{aligned} \Delta_{8,1,10+5} &:= \Delta_{10+5}^\infty (\Delta_{10+5}^0)^5, & \Delta_{8,2,10+5} &:= (\Delta_{10+5}^\infty)^5 \Delta_{10+5}^0, \\ \Delta_{8,3,10+5} &:= (\Delta_{10+5}^\infty)^2 (\Delta_{10+5}^0)^4, & \Delta_{8,4,10+5} &:= (\Delta_{10+5}^\infty)^4 (\Delta_{10+5}^0)^2. \end{aligned}$$

Now, we have

$$M_k(\Gamma_0(10) + 5) = \mathbb{C}E_{k,10+5}^\infty \oplus \mathbb{C}E_{k,10+5}^0 \oplus S_k(\Gamma_0(10) + 5),$$

$$S_k(\Gamma_0(10) + 5) = (\mathbb{C}\Delta_{8,1,10+5} \oplus \mathbb{C}\Delta_{8,2,10+5} \oplus \mathbb{C}\Delta_{8,3,10+5} \oplus \mathbb{C}\Delta_{8,4,10+5} \oplus \mathbb{C}(\Delta_{10})^3)M_{k-8}(\Gamma_0(10) + 5)$$

for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(10) + 5) = E_{2,10+5}' M_{4n}(\Gamma_0(10) + 5)$ and

$$\begin{aligned} M_{8n}(\Gamma_0(10) + 5) &= \mathbb{C}(E_{8,10+5}^\infty)^n \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} \Delta_{8,1,10+5} \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} \Delta_{8,3,10+5} \\ &\quad \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,3,10+5} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10+5}^0)^n \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} \Delta_{8,2,10+5} \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} \Delta_{8,4,10+5} \\ &\quad \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,4,10+5} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}, \\ M_{8n+4}(\Gamma_0(10) + 5) &= E_{4,10+5}^\infty (\mathbb{C}(E_{8,10+5}^\infty)^n \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} \Delta_{8,1,10+5} \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} \Delta_{8,3,10+5} \\ &\quad \oplus \mathbb{C}(E_{8,10+5}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,3,10+5} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus E_{4,10+5}^0 (\mathbb{C}(E_{8,10+5}^0)^n \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} \Delta_{8,2,10+5} \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} \Delta_{8,4,10+5} \\ &\quad \oplus \mathbb{C}(E_{8,10+5}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,4,10+5} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus (\mathbb{C}\Delta_{10+5}^\infty (\Delta_{10+5}^0)^2 \oplus \mathbb{C}(\Delta_{10+5}^\infty)^2 \Delta_{10+5}^0) (\Delta_{10})^{3n}. \end{aligned}$$

Furthermore, we can write

$$M_{4n}(\Gamma_0(10) + 5) = \mathbb{C}(\Delta_{10+5}^\infty)^{3n} \oplus \mathbb{C}(\Delta_{10+5}^\infty)^{3n-1} \Delta_{10+5}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{10+5}^0)^{3n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(10) + 5$:

$$(212) \quad J_{10+5} := \Delta_{10+5}^0 / \Delta_{10+5}^\infty (= \eta^4(z) \eta^{-4}(2z) \eta^4(5z) \eta^{-4}(10z)) = \frac{1}{q} - 4 + 6q - 8q^2 + 17q^3 - \cdots,$$

where $v_\infty(J_{10+5}) = -1$ and $v_0(J_{10+5}) = 1$. Then, we have

$$(213) \quad J_{10+5} : \partial \mathbb{F}_{10+5} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-6 - 2\sqrt{5}, 0] \subset \mathbb{R}.$$

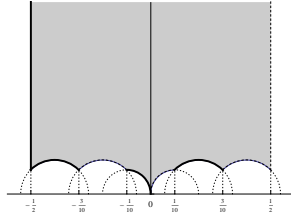
10.4. $\Gamma_0(10) + 2$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(10) + 2$ as follows:

$$(214) \quad \begin{aligned} \mathbb{F}_{10+2} &= \left\{ |z + 2/5| \geq 1/(5\sqrt{2}), -1/2 \leq \operatorname{Re}(z) \leq -3/10 \right\} \cup \left\{ |z + 1/5| > 1/(5\sqrt{2}), -3/10 < \operatorname{Re}(z) < -1/10 \right\} \\ &\quad \cup \left\{ |z + 1/10| \geq 1/10, 0 \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z - 1/10| > 1/10, 0 < \operatorname{Re}(z) < 1/10 \right\} \\ &\quad \cup \left\{ |z - 1/5| \geq 1/(5\sqrt{2}), 1/10 \leq \operatorname{Re}(z) \leq 3/10 \right\} \cup \left\{ |z - 2/5| > 1/(5\sqrt{2}), 3/10 < \operatorname{Re}(z) < 1/2 \right\}, \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 10 & -1 \end{pmatrix} : (e^{i\theta} + 1)/10 \rightarrow (e^{i(\pi-\theta)} - 1)/10$, $\begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} W_{10,5} : e^{i\theta}/(5\sqrt{2}) - 1/5 \rightarrow e^{i(\pi-\theta)}/(5\sqrt{2}) - 2/5$, and $\begin{pmatrix} 9 & 2 \\ 40 & 9 \end{pmatrix} W_{10,5} : e^{i\theta}/(5\sqrt{2}) + 2/5 \rightarrow e^{i(\pi-\theta)}/(5\sqrt{2}) + 1/5$. Then, we have

$$(215) \quad \Gamma_0(10) + 2 = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{10,2}, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 2 \\ 40 & 9 \end{pmatrix} \rangle.$$

FIGURE 33. $\Gamma_0(10) + 2$

Valence formula. The cusps of $\Gamma_0(10) + 2$ are ∞ and 0, and the elliptic points are $\rho_{10,1} = -3/10 + i/10$ and $\rho_{10,5} = 3/10 + i/10$. Let f be a modular function of weight k for $\Gamma_0(10) + 2$, which is not identically zero. We have

$$(216) \quad v_\infty(f) + v_0(f) + \frac{1}{4}v_{\rho_{10,1}}(f) + \frac{1}{4}v_{\rho_{10,5}}(f) + \sum_{\substack{p \in \Gamma_0(10)+2 \setminus \mathbb{H} \\ p \neq \rho_{10,1}, \rho_{10,5}}} v_p(f) = \frac{3k}{4}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{10,1}$ (resp. $\rho_{10,5}$) is $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}, \pm W_{10,2}, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} W_{10,2}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 3 & -1 \\ 10 & -3 \end{pmatrix}, \pm \begin{pmatrix} 13 & 3 \\ 30 & 7 \end{pmatrix} W_{10,2}, \pm \begin{pmatrix} 9 & 2 \\ 40 & 9 \end{pmatrix} W_{10,2}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(10) + 2$:

$$(217) \quad E_{k,10+2}^\infty(z) := \frac{2^{k/2}5^k E_k(10z) + 5^k E_k(5z) - 2^{k/2} E_k(2z) - E_k(z)}{(5^k - 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 10n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{10}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(10) + 2$:

$$(218) \quad E_{k,10+2}^0(z) := \frac{-5^{k/2}(2^{k/2} E_k(10z) + E_k(5z) - 2^{k/2} E_k(2z) - E_k(z))}{(5^k - 1)(2^{k/2} + 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1}(\Gamma_0(10) + 2)\gamma_0 = \Gamma_0(10) + 2$.

The space of modular forms. We define

$$\Delta_{10+2}^\infty := \Delta_{10}^\infty \Delta_{10}^{-1/5}, \quad \Delta_{10+2}^0 := \Delta_{10}^0 \Delta_{10}^{-1/2},$$

which are 2nd semimodular forms for $\Gamma_0(10) + 2$ of weight 2. Furthermore, we define

$$\begin{aligned} \Delta_{8,1,10+2} &:= \Delta_{10+2}^\infty (\Delta_{10+2}^0)^5, & \Delta_{8,2,10+2} &:= (\Delta_{10+2}^\infty)^5 \Delta_{10+2}^0, \\ \Delta_{8,3,10+2} &:= (\Delta_{10+2}^\infty)^2 (\Delta_{10+2}^0)^4, & \Delta_{8,4,10+2} &:= (\Delta_{10+2}^\infty)^4 (\Delta_{10+2}^0)^2. \end{aligned}$$

Now, we have

$$M_k(\Gamma_0(10) + 2) = \mathbb{C}E_{k,10+2}^\infty \oplus \mathbb{C}E_{k,10+2}^0 \oplus S_k(\Gamma_0(10) + 2),$$

$$S_k(\Gamma_0(10) + 2) = (\mathbb{C}\Delta_{8,1,10+2} \oplus \mathbb{C}\Delta_{8,2,10+2} \oplus \mathbb{C}\Delta_{8,3,10+2} \oplus \mathbb{C}\Delta_{8,4,10+2} \oplus \mathbb{C}(\Delta_{10})^3) M_{k-8}(\Gamma_0(10) + 2)$$

for every even integer $k \geq 4$. Then, we have $M_{8n+2}(\Gamma_0(10) + 2) = E_{2,10+2}' M_{8n}(\Gamma_0(10) + 2)$ and

$$\begin{aligned} M_{8n}(\Gamma_0(10) + 2) &= \mathbb{C}(E_{8,10+2}^\infty)^n \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,1,10+2} \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,3,10+2} \\ &\quad \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,3,10+2} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10+2}^0)^n \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,2,10+2} \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,4,10+2} \\ &\quad \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,4,10+2} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}, \end{aligned}$$

$$\begin{aligned}
M_{8n+4}(\Gamma_0(10) + 2) &= E_{4,10+2}^\infty (\mathbb{C}(E_{8,10+2}^\infty)^n \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,1,10+2} \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,3,10+2} \\
&\quad \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C} \Delta_{8,3,10+2} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C} (\Delta_{10})^{3n}) \\
&\quad \oplus E_{4,10+2}^0 (\mathbb{C}(E_{8,10+2}^0)^n \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,2,10+2} \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,4,10+2} \\
&\quad \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C} \Delta_{8,4,10+2} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C} (\Delta_{10})^{3n}) \\
&\quad \oplus (E_{2/3,10}')^2 (\mathbb{C}(\Delta_{10+2}^\infty)^2 \oplus \mathbb{C}(\Delta_{10+2}^0)^2 \oplus \mathbb{C} \Delta_{10}) (\Delta_{10})^{3n}, \\
M_{8n+6}(\Gamma_0(10) + 2) &= E_{6,10+2}^\infty (\mathbb{C}(E_{8,10+2}^\infty)^n \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,1,10+2} \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} \Delta_{8,3,10+2} \\
&\quad \oplus \mathbb{C}(E_{8,10+2}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C} \Delta_{8,3,10+2} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C} (\Delta_{10})^{3n}) \\
&\quad \oplus E_{6,10+2}^0 (\mathbb{C}(E_{8,10+2}^0)^n \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,2,10+2} \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} \Delta_{8,4,10+2} \\
&\quad \oplus \mathbb{C}(E_{8,10+2}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C} \Delta_{8,4,10+2} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C} (\Delta_{10})^{3n}) \\
&\quad \oplus E_{2/3,10}' (\mathbb{C}(\Delta_{10+2}^\infty)^4 \oplus \mathbb{C}(\Delta_{10+2}^0)^4 \oplus \mathbb{C}(\Delta_{10+2}^\infty)^3 \Delta_{10+2}^0 \\
&\quad \oplus \mathbb{C} \Delta_{10+2}^\infty (\Delta_{10+2}^0)^3 \oplus \mathbb{C}(\Delta_{10})^2) (\Delta_{10})^{3n}.
\end{aligned}$$

Here, we define $E_{2/3,10+2}' := \sqrt[3]{E_{2,10+2}'}$ where $v_{\rho_{10,1}}(E_{2/3,10+2}') = v_{\rho_{10,5}}(E_{2/3,10+2}') = 1$, and we can write

$$M_k(\Gamma_0(10) + 2) = E_{\bar{k},10+2}' (\mathbb{C}(\Delta_{10+2}^\infty)^n \oplus \mathbb{C}(\Delta_{10+2}^\infty)^{n-1} \Delta_{10+2}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{10+2}^0)^n),$$

where $n = \dim(M_k(\Gamma_0(10) + 2)) - 1 = \lfloor 3k/4 - 2(3k/8 - \lfloor 3k/8 \rfloor) \rfloor$, and where $E_{\bar{k},10+2}' := 1$, $(E_{2/3,10+2}')^3$, $(E_{2/3,10+2}')^2$, and $E_{2/3,10+2}'$, when $k \equiv 0, 2, 4$, and $6 \pmod{8}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(10) + 2$:

$$(219) \quad J_{10+2} := \Delta_{10+2}^0 / \Delta_{10+2}^\infty (= \eta^2(z) \eta^2(2z) \eta^{-2}(5z) \eta^{-2}(10z)) = \frac{1}{q} - 2 - 3q + 6q^2 + 2q^3 - \cdots,$$

where $v_\infty(J_{10+2}) = -1$ and $v_0(J_{10+2}) = 1$. Then, we have

$$\begin{aligned}
(220) \quad J_{10+2} : \quad &\{|z + 1/10| = 1/10, -1/10 \leq \operatorname{Re}(z) \leq 0\} \rightarrow [-1, 0] \subset \mathbb{R}, \\
&\{|z + 2/5| = 1/(5\sqrt{2}), -1/2 \leq \operatorname{Re}(z) \leq -3/10\} \rightarrow \{-3 \leq \operatorname{Re}(z) \leq -1, 0 \leq \operatorname{Im}(z) \leq 4\}, \\
&\{|z - 1/5| = 1/(5\sqrt{2}), 1/10 \leq \operatorname{Re}(z) \leq 3/10\} \rightarrow \{-3 \leq \operatorname{Re}(z) \leq -1, -4 \leq \operatorname{Im}(z) \leq 0\}.
\end{aligned}$$

Thus, J_{10+2} does not take real value on some arcs of $\partial \mathbb{F}_{10+2}$.

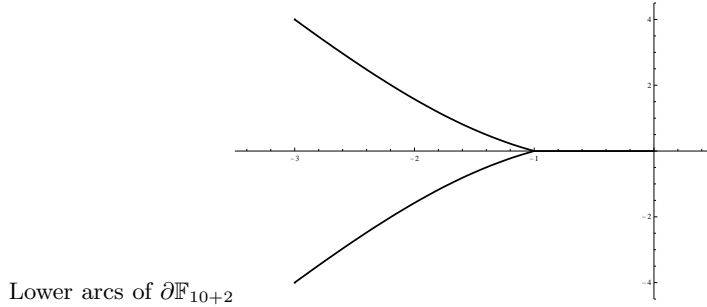


FIGURE 34. Image by J_{10+2}

10.5. $\Gamma_0(10)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(10)$ as follows:

$$\begin{aligned}
(221) \quad \mathbb{F}_{10} = &\{|z + 9/20| \geq 1/20, -1/2 \leq \operatorname{Re}(z) < -2/5\} \cup \{|z + 3/10| \geq 1/10, -2/5 \leq \operatorname{Re}(z) \leq -3/10\} \\
&\cup \{|z + 3/10| > 1/10, -3/10 < \operatorname{Re}(z) < -1/5\} \cup \{|z + 1/10| \geq 1/10, -1/5 \leq \operatorname{Re}(z) \leq 0\} \\
&\cup \{|z - 1/10| > 1/10, 0 < \operatorname{Re}(z) < 1/5\} \cup \{|z - 3/10| \leq 1/10, 1/5 \leq \operatorname{Re}(z) \leq 3/10\} \\
&\cup \{|z - 3/10| > 1/10, 3/10 < \operatorname{Re}(z) \leq 2/5\} \cup \{|z - 9/20| > 1/20, 2/5 < \operatorname{Re}(z) < 1/2\},
\end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 10 & -1 \end{pmatrix} : (e^{i\theta} + 1)/10 \rightarrow (e^{i(\pi-\theta)} - 1)/10$, $\begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix} : (e^{i\theta} - 3)/10 \rightarrow (e^{i(\pi-\theta)} - 3)/10$, $\begin{pmatrix} 3 & -1 \\ 10 & -3 \end{pmatrix} : (e^{i\theta} + 3)/10 \rightarrow (e^{i(\pi-\theta)} + 3)/10$, and $\begin{pmatrix} -9 & 4 \\ 20 & -9 \end{pmatrix} : (e^{i\theta} + 9)/20 \rightarrow (e^{i(\pi-\theta)} - 9)/20$. Then, we have

$$(222) \quad \Gamma_0(10) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 10 & 1 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 10 & -3 \end{pmatrix}, \begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix} \rangle.$$

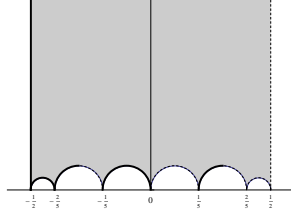


FIGURE 35. $\Gamma_0(10)$

Valence formula. The cusps of $\Gamma_0(10)$ are ∞ , 0 , $-1/2$, and $-1/5$. Let f be a modular function of weight k for $\Gamma_0(10)$, which is not identically zero. We have

$$(223) \quad v_\infty(f) + v_0(f) + v_{-1/2}(f) + v_{-1/5}(f) + \frac{1}{2}v_{\rho_{10,1}}(f) + \frac{1}{2}v_{\rho_{10,5}}(f) + \sum_{\substack{p \in \Gamma_0(10) \backslash \mathbb{H} \\ p \neq \rho_{10,1}, \rho_{10,5}}} v_p(f) = \frac{3k}{2}.$$

Furthermore, the stabilizer of the elliptic point $\rho_{10,1}$ (resp. $\rho_{10,5}$) is $\{\pm I, \pm \begin{pmatrix} -3 & -1 \\ 10 & 3 \end{pmatrix}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 3 & -1 \\ 10 & -3 \end{pmatrix}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(10)$:

$$(224) \quad E_{k,10}^\infty(z) := \frac{10^k E_k(10z) - 5^k E_k(5z) - 2^k E_k(2z) + E_k(z)}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp 0 . We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 10n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{10}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(10)$:

$$(225) \quad E_{k,10}^0(z) := \frac{10^{k/2}(E_k(10z) - E_k(5z) - E_k(2z) + E_k(z))}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(10) \gamma_0 = \Gamma_0(10)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 10n+1 & 5n \\ -20n & -10n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{10,5}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(10)$:

$$(226) \quad E_{k,10}^{-1/2}(z) := \frac{-5^{k/2}(2^k E_k(10z) - E_k(5z) - 2^k E_k(2z) + E_k(z))}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0(10) \gamma_{-1/2} = \Gamma_0(10)$.

For the cusp $-1/5$. We have $\Gamma_{-1/5} = \{\pm \begin{pmatrix} 10n+1 & 2n \\ -50n & -10n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/5} = W_{10,2}$, and we have the Eisenstein series for the cusp $-1/5$ associated with $\Gamma_0(10)$:

$$(227) \quad E_{k,10}^{-1/5}(z) := \frac{-2^{k/2}(5^k E_k(10z) - 5^k E_k(5z) - E_k(2z) + E_k(z))}{(5^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_{-1/5}^{-1} \Gamma_0(10) \gamma_{-1/5} = \Gamma_0(10)$.

The space of modular forms. We define

$$\begin{aligned}\Delta_{8,1,10} &:= (E_{2/3,10}')^2 (\Delta_{10}^0)^2 (\Delta_{10}^{-1/2})^2 (\Delta_{10}^{-1/5})^2 \Delta_{10}, & \Delta_{8,2,10} &:= (E_{2/3,10}')^2 (\Delta_{10}^\infty)^2 (\Delta_{10}^{-1/2})^2 (\Delta_{10}^{-1/5})^2 \Delta_{10}, \\ \Delta_{8,3,10} &:= (E_{2/3,10}')^2 (\Delta_{10}^\infty)^2 (\Delta_{10}^0)^2 (\Delta_{10}^{-1/5})^2 \Delta_{10}, & \Delta_{8,4,10} &:= (E_{2/3,10}')^2 (\Delta_{10}^\infty)^2 (\Delta_{10}^0)^2 (\Delta_{10}^{-1/2})^2 \Delta_{10}, \\ \Delta_{8,5,10} &:= (\Delta_{10}^0)^2 \Delta_{10}^{-1/2} \Delta_{10}^{-1/5} (\Delta_{10})^2, & \Delta_{8,6,10} &:= (\Delta_{10}^\infty)^2 \Delta_{10}^{-1/2} \Delta_{10}^{-1/5} (\Delta_{10})^2, \\ \Delta_{8,7,10} &:= \Delta_{10}^\infty \Delta_{10}^0 (\Delta_{10}^{-1/5})^2 (\Delta_{10})^2, & \Delta_{8,8,10} &:= \Delta_{10}^\infty \Delta_{10}^0 (\Delta_{10}^{-1/2})^2 (\Delta_{10})^2.\end{aligned}$$

Now, we have

$$\begin{aligned}M_k(\Gamma_0(10)) &= \mathbb{C}E_{k,10}^\infty \oplus \mathbb{C}E_{k,10}^0 \oplus \mathbb{C}E_{k,10}^{-1/2} \oplus \mathbb{C}E_{k,10}^{-1/5} \oplus S_k(\Gamma_0(10)), \\ S_k(\Gamma_0(10)) &= (\mathbb{C}\Delta_{8,1,10} \oplus \mathbb{C}\Delta_{8,2,10} \oplus \mathbb{C}\Delta_{8,3,10} \oplus \mathbb{C}\Delta_{8,4,10} \oplus \mathbb{C}\Delta_{8,5,10} \\ &\quad \oplus \mathbb{C}\Delta_{8,6,10} \oplus \mathbb{C}\Delta_{8,7,10} \oplus \mathbb{C}\Delta_{8,8,10} \oplus \mathbb{C}(\Delta_{10})^3) M_{k-8}(\Gamma_0(10))\end{aligned}$$

for every even integer $k \geq 4$. Then, we have $M_{8n+2}(\Gamma_0(10)) = E_{2,10+2}' M_{8n}(\Gamma_0(10)) \oplus \mathbb{C}E_{2,10+10}' (\Delta_{10})^{3n} \oplus \mathbb{C}E_{2,10+5}' (\Delta_{10})^{3n}$, $M_{8n+6}(\Gamma_0(10)) = E_{2,10+2}' M_{8n+4}(\Gamma_0(10)) \oplus \mathbb{C}(E_{2,10+10}')^3 (\Delta_{10})^{3n} \oplus \mathbb{C}(E_{2,10+5}')^3 (\Delta_{10})^{3n}$, and

$$\begin{aligned}M_{8n}(\Gamma_0(10)) &= \mathbb{C}(E_{8,10}^\infty)^n \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} \Delta_{8,1,10} \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} \Delta_{8,5,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,5,10} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10}^0)^n \oplus \mathbb{C}(E_{8,10}^0)^{n-1} \Delta_{8,2,10} \oplus \mathbb{C}(E_{8,10}^0)^{n-1} \Delta_{8,6,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,6,10} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/2})^n \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} \Delta_{8,3,10} \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} \Delta_{8,7,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,7,10} (\Delta_{10})^{3(n-1)} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/5})^n \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} \Delta_{8,4,10} \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} \Delta_{8,8,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,8,10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}, \\ M_{8n+4}(\Gamma_0(10)) &= E_{4,10}^\infty (\mathbb{C}(E_{8,10}^\infty)^n \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} \Delta_{8,1,10} \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} \Delta_{8,5,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^\infty)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,5,10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus E_{4,10}^0 (\mathbb{C}(E_{8,10}^0)^n \oplus \mathbb{C}(E_{8,10}^0)^{n-1} \Delta_{8,2,10} \oplus \mathbb{C}(E_{8,10}^0)^{n-1} \Delta_{8,6,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^0)^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,6,10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus E_{4,10}^{-1/2} (\mathbb{C}(E_{8,10}^{-1/2})^n \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} \Delta_{8,3,10} \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} \Delta_{8,7,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/2})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,7,10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus E_{4,10}^{-1/5} (\mathbb{C}(E_{8,10}^{-1/5})^n \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} \Delta_{8,4,10} \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} \Delta_{8,8,10} \\ &\quad \oplus \mathbb{C}(E_{8,10}^{-1/5})^{n-1} (\Delta_{10})^3 \oplus \cdots \oplus \mathbb{C}\Delta_{8,8,10} (\Delta_{10})^{3(n-1)} \oplus \mathbb{C}(\Delta_{10})^{3n}) \\ &\quad \oplus (\mathbb{C}\Delta_{10}^\infty \Delta_{10}^0 \oplus \mathbb{C}\Delta_{10}^0 \Delta_{10}^{-1/2} \oplus \mathbb{C}\Delta_{10}^{-1/2} \Delta_{10}^{-1/5}) (\Delta_{10})^{3n+1}.\end{aligned}$$

Furthermore, we can write

$$M_k(\Gamma_0(10)) = E_{k,10}' (\mathbb{C}(\Delta_{10}^\infty)^n \oplus \mathbb{C}(\Delta_{10}^\infty)^{n-1} \Delta_{10}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{10}^0)^n),$$

where $n = \dim(M_k(\Gamma_0(10))) - 1 = \lfloor 3k/2 - 2(k/4 - \lfloor k/4 \rfloor) \rfloor$, and where $E_{k,10}' := 1$ and $E_{2/3,10+2}'$, when $k \equiv 0$ and $2 \pmod{4}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(10)$:

$$(228) \quad J_{10} := \Delta_{10}^0 / \Delta_{10}^\infty (= \eta^3(z) \eta^{-1}(2z) \eta(5z) \eta^{-3}(10z)) = \frac{1}{q} - 3 + q + 2q^2 + 2q^3 - \cdots,$$

where $v_\infty(J_{10}) = -1$ and $v_0(J_{10}) = 1$. Then, we have

$$(229) \quad \begin{aligned} J_{10} : \quad & \{|z + 1/10| = 1/10, -1/10 \leq \operatorname{Re}(z) \leq 0\} \rightarrow [-4, 0] \subset \mathbb{R}, \\ & \{|z + 3/10| = 1/10, -2/5 \leq \operatorname{Re}(z) \leq -3/10\} \rightarrow \{-4 \leq \operatorname{Re}(z) < -3.8, 0 \leq \operatorname{Im}(z) \leq 2\}, \\ & \{|z - 3/10| = 1/10, 1/5 \leq \operatorname{Re}(z) \leq 3/10\} \rightarrow \{-4 \leq \operatorname{Re}(z) < -3.8, -2 \leq \operatorname{Im}(z) \leq 0\}, \\ & \{|z + 9/20| = 1/20, -1/2 \leq \operatorname{Re}(z) \leq -2/5\} \rightarrow [-5, -4] \subset \mathbb{R}. \end{aligned}$$

Thus, J_{10} does not take real value on some arcs of $\partial\mathbb{F}_{10}$.

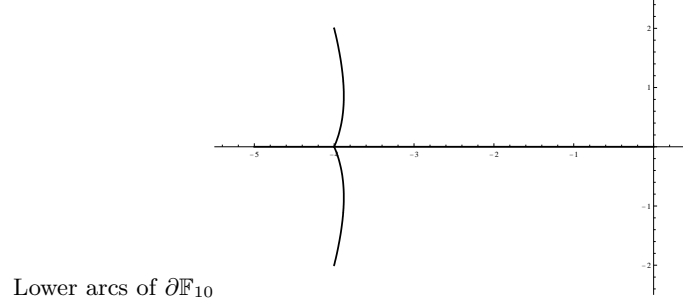


FIGURE 36. Image by J_{10}

11. LEVEL 11

We have $\Gamma_0(11)+ = \Gamma_0^*(11)$ and $\Gamma_0(11)- = \Gamma_0(11)$, but $\Gamma_0(11)$ is of genus 1.

We have $W_{11} = \begin{pmatrix} 0 & -1/\sqrt{11} \\ \sqrt{11} & 0 \end{pmatrix}$, and denote $\rho_{11,1} := -1/2 + i/(2\sqrt{11})$, $\rho_{11,2} := -1/3 + i/(3\sqrt{11})$, and $\rho_{11,3} := 1/3 + i/(3\sqrt{11})$. We define

$$(230) \quad \begin{aligned} \Delta_{11}^\infty(z) &:= \sqrt[5]{\eta^{11}(11z)/\eta(z)}, & \Delta_{11}^0(z) &:= \sqrt[5]{\eta^{11}(z)/\eta(11z)}, \\ \Delta_{11}(z) &:= \Delta_{11}^\infty(z)\Delta_{11}^0(z) = \eta^2(z)\eta^2(11z), \end{aligned}$$

where Δ_{11}^∞ and Δ_{11}^0 are 2nd semimodular forms for $\Gamma_0(11)$ of weight 1 such that $v_\infty(\Delta_{11}^\infty) = v_0(\Delta_{11}^0) = 1$, and Δ_{11} is a cusp form for $\Gamma_0(11)$ and 2nd semimodular form for $\Gamma_0^*(11)$ of weight 2. Furthermore, we define

$$(231) \quad E_{2,11}'(z) := (11E_2(11z) - E_2(z))/10,$$

which is a modular form for $\Gamma_0(11)$ and 2nd semimodular form for $\Gamma_0^*(11)$ of weight 2.

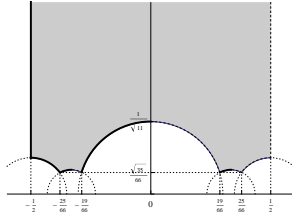
11.1. $\Gamma_0^*(11)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(11)$ as follows:

$$(232) \quad \begin{aligned} \mathbb{F}_{11+} = & \left\{ |z + 1/2| \geq 1/(2\sqrt{11}), -1/2 \leq \operatorname{Re}(z) < -25/66 \right\} \cup \left\{ |z + 1/3| \geq 1/(3\sqrt{11}), -25/66 \leq \operatorname{Re}(z) \leq -1/3 \right\} \\ & \cup \left\{ |z + 1/3| > 1/(3\sqrt{11}), -1/3 < \operatorname{Re}(z) < -19/66 \right\} \cup \left\{ |z| \geq 1/\sqrt{11}, -19/66 \leq \operatorname{Re}(z) \leq 0 \right\} \\ & \cup \left\{ |z| > 1/\sqrt{11}, 0 < \operatorname{Re}(z) < 19/66 \right\} \cup \left\{ |z - 1/3| \geq 1/(3\sqrt{11}), 19/66 \leq \operatorname{Re}(z) \leq 1/3 \right\} \\ & \cup \left\{ |z - 1/3| > 1/(3\sqrt{11}), 1/3 < \operatorname{Re}(z) \leq 25/66 \right\} \cup \left\{ |z - 1/2| > 1/(2\sqrt{11}), 25/66 < \operatorname{Re}(z) < 1/2 \right\}. \end{aligned}$$

where $W_{11} : e^{i\theta}/\sqrt{11} \rightarrow e^{i(\pi-\theta)}/\sqrt{11}$, $\begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix} W_{11} : e^{i\theta}/(2\sqrt{11}) + 1/2 \rightarrow e^{i(\pi-\theta)}/(2\sqrt{11}) - 1/2$, $\begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix} W_{11} : e^{i\theta}/(3\sqrt{11}) - 1/3 \rightarrow e^{i(\pi-\theta)}/(3\sqrt{11}) - 1/3$, and $\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} W_{11} : e^{i\theta}/(3\sqrt{11}) + 1/3 \rightarrow e^{i(\pi-\theta)}/(3\sqrt{11}) + 1/3$. Then, we have

$$(233) \quad \Gamma_0^*(11) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{11}, \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \rangle.$$

FIGURE 37. $\Gamma_0^*(11)$

Valence formula. The cusp of $\Gamma_0^*(11)$ is ∞ , and the elliptic points are $i/\sqrt{11}$, $\rho_{11,1} = -1/2 + i\sqrt{11}/10$, $\rho_{11,2} = -1/3 + i/(3\sqrt{11})$, and $\rho_{11,3} = 1/3 + i/(3\sqrt{11})$. Let f be a modular function of weight k for $\Gamma_0^*(11)$, which is not identically zero. We have

$$(234) \quad v_\infty(f) + \frac{1}{2}v_{i/\sqrt{11}}(f) + \frac{1}{2}v_{\rho_{11,1}}(f) + \frac{1}{2}v_{\rho_{11,2}}(f) + \frac{1}{2}v_{\rho_{11,3}}(f) + \sum_{\substack{p \in \Gamma_0^*(11) \backslash \mathbb{H} \\ p \neq i/\sqrt{11}, \rho_{11,1}, \rho_{11,2}, \rho_{11,3}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $i/\sqrt{11}$ (resp. $\rho_{11,1}$, $\rho_{11,2}$, $\rho_{11,3}$) is $\{\pm I, \pm W_{11}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 6 & -1 \\ -11 & 2 \end{pmatrix} W_{11}\}$, $\{\pm I, \pm \begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix} W_{11}\}$, $\{\pm I, \pm \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} W_{11}\}$)

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series associated with $\Gamma_0^*(11)$:

$$(235) \quad E_{k,11+}(z) := \frac{11^{k/2}E_k(11z) + E_k(z)}{11^{k/2} + 1} \quad \text{for } k \geq 4.$$

The space of modular forms. We define the following functions:

$$E_{4,11}' := (-1525(E_{2,11}')^2 + 4320E_{2,11}'\Delta_{11} + 2016(\Delta_{11})^2 + 3050E_{4,11}^\infty)/1525,$$

where $E_{4,11}^\infty$ is the Eisenstein series of weight 4 for the cusp ∞ for $\Gamma_0(11)$. Then, $E_{4,11}'$ is a 2nd semimodular form for $\Gamma_0^*(11)$ of weight 4 such that $v_{i/\sqrt{11}}(E_{4,11}') = v_{\rho_{11,1}}(E_{4,11}') = v_{\rho_{11,2}}(E_{4,11}') = v_{\rho_{11,3}}(E_{4,11}') = 1$.

Let k be an even integer $k \geq 4$. We have $M_k(\Gamma_0^*(11)) = \mathbb{C}E_{k,11+} \oplus S_k(\Gamma_0^*(11))$ and $S_k(\Gamma_0^*(11)) = (\mathbb{C}E_{2,11}'\Delta_{11} \oplus \mathbb{C}(\Delta_{11})^2)M_{k-4}(\Gamma_0^*(11))$. Then, we have

$$\begin{aligned} M_{4n}(\Gamma_0^*(11)) &= \mathbb{C}(E_{2,11}')^{2n} \oplus \mathbb{C}(E_{2,11}')^{2n-1}\Delta_{11} \oplus \cdots \oplus \mathbb{C}(\Delta_{11})^{2n}, \\ M_{4n+6}(\Gamma_0^*(11)) &= E_{4,11}'(\mathbb{C}(E_{2,11}')^{2n+1} \oplus \mathbb{C}(E_{2,11}')^{2n}\Delta_{11} \oplus \cdots \oplus \mathbb{C}(\Delta_{11})^{2n+1}). \end{aligned}$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(11)$:

$$(236) \quad J_{11+} := E_{2,11}'/\Delta_{11} = \frac{1}{q} + \frac{22}{5} + 17q + 46q^2 + 116q^3 + \cdots,$$

where $v_\infty(J_{11+}) = -1$. Then, we have

$$(237) \quad \begin{aligned} J_{11+} : \quad & \left\{ |z| = 1/\sqrt{11}, -19/66 \leq \operatorname{Re}(z) \leq 0 \right\} \rightarrow [22/5 - 2\sqrt{5}, 15.22750...] \subset \mathbb{R}, \\ & \left\{ |z + 1/3| = 1/(3\sqrt{11}), -25/66 \leq \operatorname{Re}(z) \leq -1/3 \right\} \rightarrow \{22/5 - 2\sqrt{5} \leq \operatorname{Re}(z) \leq -0.013750..., 0 \leq \operatorname{Im}(z) \leq 0.31397...\}, \\ & \left\{ |z - 1/3| = 1/(3\sqrt{11}), 19/66 \leq \operatorname{Re}(z) \leq 1/3 \right\} \rightarrow \{22/5 - 2\sqrt{5} \leq \operatorname{Re}(z) \leq -0.013750..., -0.31397... \leq \operatorname{Im}(z) \leq 0\} \\ & \left\{ |z + 1/2| = 1/(2\sqrt{11}), -1/2 \leq \operatorname{Re}(z) \leq -25/66 \right\} \rightarrow [-8/5, 22/5 - 2\sqrt{5}] \subset \mathbb{R}. \end{aligned}$$

Thus, J_{11+} does not take real value on some arcs of $\partial\mathbb{F}_{11+}$.

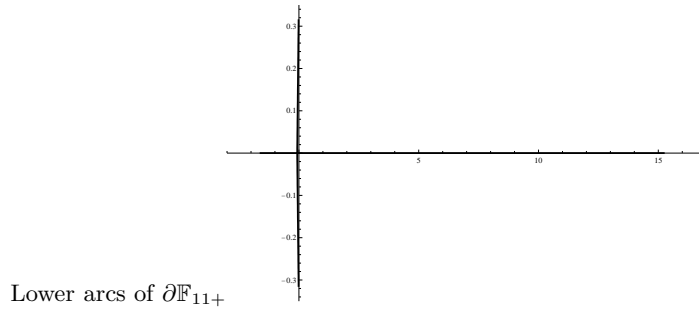


FIGURE 38. Image by J_{11+}

11.2. $\Gamma_0(11)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(11)$ as follows:

$$(238) \quad \begin{aligned} \mathbb{F}_{11} = & \{ |z + 5/11| \geq 1/11, -1/2 \leq \operatorname{Re}(z) < -9/22 \} \cup \{ |z + 4/11| \geq 1/11, -9/22 \leq \operatorname{Re}(z) < -7/22 \} \\ & \{ |z + 3/11| \geq 1/11, -7/22 \leq \operatorname{Re}(z) \leq -5/22 \} \cup \{ |z + 2/11| > 1/11, -5/22 < \operatorname{Re}(z) < -3/22 \} \\ & \cup \{ |z + 1/11| \geq 1/11, -3/22 \leq \operatorname{Re}(z) \leq 0 \} \cup \{ |z - 1/11| > 1/11, 0 < \operatorname{Re}(z) < 3/22 \} \\ & \cup \{ |z - 2/11| \geq 1/11, 3/22 \leq \operatorname{Re}(z) \leq 5/22 \} \cup \{ |z - 3/11| > 1/11, 5/22 < \operatorname{Re}(z) < 7/22 \} \\ & \cup \{ |z - 4/11| > 1/11, 7/22 \leq \operatorname{Re}(z) < 9/22 \} \cup \{ |z - 5/11| > 1/11, 9/22 \leq \operatorname{Re}(z) < 1/2 \}. \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 11 & -1 \end{pmatrix} : (e^{i\theta} + 1)/11 \rightarrow (e^{i(\pi-\theta)} - 1)/11$, $\begin{pmatrix} -5 & -1 \\ 11 & 2 \end{pmatrix} : (e^{i\theta} - 2)/11 \rightarrow (e^{i(\pi-\theta)} - 5)/11$, $\begin{pmatrix} 2 & -1 \\ 11 & -5 \end{pmatrix} : (e^{i\theta} + 5)/11 \rightarrow (e^{i(\pi-\theta)} + 2)/11$, $\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} : (e^{i\theta} + 4)/11 \rightarrow (e^{i(\pi-\theta)} - 3)/11$, and $\begin{pmatrix} 4 & -1 \\ -11 & 3 \end{pmatrix} : (e^{i\theta} + 3)/11 \rightarrow (e^{i(\pi-\theta)} - 4)/11$. Then, we have

$$(239) \quad \Gamma_0(11) = \langle -I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \rangle.$$

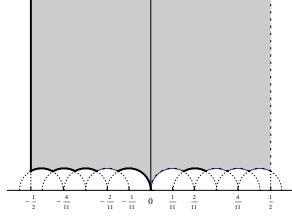


FIGURE 39. $\Gamma_0(11)$

Valence formula. The cusps of $\Gamma_0(11)$ are ∞ and 0. Let f be a modular function of weight k for $\Gamma_0(11)$, which is not identically zero. We have

$$(240) \quad v_\infty(f) + v_0(f) + \sum_{p \in \Gamma_0(11) \backslash \mathbb{H}} v_p(f) = k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(11)$:

$$(241) \quad E_{k,11}^\infty(z) := \frac{11^k E_k(11z) - E_k(z)}{11^k - 1} \quad \text{for } k \geq 4.$$

For the cusp 0. We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 11n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{11}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(11)$:

$$(242) \quad E_{k,11}^0(z) := \frac{-11^{k/2}(E_k(11z) - E_k(z))}{11^k - 1} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1} \Gamma_0(11) \gamma_0 = \Gamma_0(11)$.

12. LEVEL 12

We have $\Gamma_0(12)+$, $\Gamma_0(12) + 12 = \Gamma_0^*(12)$, $\Gamma_0(12) + 4$, $\Gamma_0(12) + 3$, and $\Gamma_0(12)- = \Gamma_0(12)$.

We have $W_{12} = \begin{pmatrix} 0 & -1/(2\sqrt{3}) \\ 2\sqrt{3} & 0 \end{pmatrix}$, $W_{12,3} := \begin{pmatrix} -\sqrt{3} & -1/\sqrt{3} \\ 4\sqrt{3} & \sqrt{3} \end{pmatrix}$, $W_{12,4} := \begin{pmatrix} -2 & 1/2 \\ 6 & -2 \end{pmatrix}$, $W_{12-,2} := \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$, $W_{12+,2} := \begin{pmatrix} -1/\sqrt{2} & -1/(2\sqrt{2}) \\ 3\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, $W_{12-,6} := \begin{pmatrix} -\sqrt{3} & -2/\sqrt{3} \\ 2\sqrt{3} & \sqrt{3} \end{pmatrix}$, and $W_{12+,6} := \begin{pmatrix} -\sqrt{6}/2 & -5/(2\sqrt{6}) \\ \sqrt{6} & \sqrt{6}/2 \end{pmatrix}$, and we denote $\rho_{12,1} := -1/4 + i/(4\sqrt{3})$, $\rho_{12,2} := -2/7 + i/(14\sqrt{3})$, and $\rho_{12,3} := 1/4 + i/(4\sqrt{3})$. We define

$$(243) \quad \begin{aligned} \Delta_{12}^\infty(z) &:= \sqrt{\eta(2z)\eta^{-2}(4z)\eta^{-3}(6z)\eta^6(12z)}, & \Delta_{12}^0(z) &:= \sqrt{\eta^6(z)\eta^{-3}(2z)\eta^{-2}(3z)\eta(6z)}, \\ \Delta_{12}^{-1/3}(z) &:= \sqrt{\eta^{-2}(z)\eta(2z)\eta^6(3z)\eta^{-3}(6z)}, & \Delta_{12}^{-1/4}(z) &:= \sqrt{\eta^{-3}(2z)\eta^6(4z)\eta(6z)\eta^{-2}(12z)}, \\ \Delta_{12}^{-1/2}(z) &:= \sqrt{\eta^{-6}(z)\eta^{15}(2z)\eta^2(3z)\eta^{-6}(4z)\eta^{-5}(6z)\eta^2(12z)}, \\ \Delta_{12}^{-1/6}(z) &:= \sqrt{\eta^2(z)\eta^{-5}(2z)\eta^{-6}(3z)\eta^2(4z)\eta^{15}(6z)\eta^{-2}(12z)}, \\ \Delta_{12} &:= \Delta_{12}^\infty \Delta_{12}^0 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} \Delta_{12}^{-1/2} \Delta_{12}^{-1/6}, & \Delta_{12+} &:= \Delta_{12}^\infty \Delta_{12}^0 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} (\Delta_{12}^{-1/2})^2 (\Delta_{12}^{-1/6})^2, \end{aligned}$$

where Δ_{12}^∞ , Δ_{12}^0 , $\Delta_{12}^{-1/3}$, $\Delta_{12}^{-1/4}$, $\Delta_{12}^{-1/2}$, and $\Delta_{12}^{-1/6}$ are 4th semimodular forms for $\Gamma_0(12)$ of weight 1 such that $v_\infty(\Delta_{12}^\infty) = v_0(\Delta_{12}^0) = v_{-1/3}(\Delta_{12}^{-1/3}) = v_{-1/4}(\Delta_{12}^{-1/4}) = v_{-1/2}(\Delta_{12}^{-1/2}) = v_{-1/6}(\Delta_{12}^{-1/6}) = 1$. Furthermore, we define

$$(244) \quad \begin{aligned} E_{2,12+}'(z) &:= (12E_2(12z) - 12E_2(6z) + 4E_2(4z) + 3E_2(3z) - 4E_2(2z) + E_2(z))/4, \\ E_{1,12+12}'(z) &:= \sqrt{-(12E_2(12z) - 18E_2(6z) - 4E_2(4z) + 3E_2(3z) + 6E_2(2z) - E_2(z))/2}, \\ E_{1,12+3}'(z) &:= \sqrt{(3E_2(6z) - E_2(z))/2}, \end{aligned}$$

where $E_{2,12+}'$ is a modular form of weight 2 and $E_{1,12+12}'$, $E_{1,12+3}'$ are 2nd semimodular forms of weight 1 for $\Gamma_0(12)$, and we have $v_{i/(2\sqrt{3})}(E_{2,12+}') = v_{\rho_{12,1}}(E_{2,12+}') = v_{\rho_{12,2}}(E_{2,12+}') = v_{\rho_{12,3}}(E_{2,12+}') = 1$, $v_{i/(2\sqrt{3})}(E_{1,12+12}') = v_{\rho_{12,2}}(E_{1,12+12}') = 1$, and $v_{\rho_{12,1}}(E_{1,12+3}') = v_{\rho_{12,3}}(E_{1,12+3}') = 1$.

12.1. $\Gamma_0(12)+$.

We have

$$\Gamma_0(12)+ = \Gamma_0(12) + 3, 4, 12 = \Gamma_0(12) \cup \Gamma_0(12)W_{12,3} \cup \Gamma_0(12)W_{12,4} \cup \Gamma_0(12)W_{12},$$

and $\Gamma_0(12)+ = T_{1/2}^{-1}(\Gamma_0(6) + 3)T_{1/2}$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(12)+$ as follows:

$$(245) \quad \begin{aligned} \mathbb{F}_{12+} &= \{|z + 1/3| \geq 1/6, -1/2 \leq \operatorname{Re}(z) \leq -1/4\} \cup \{|z| \geq 1/(2\sqrt{3}), -1/4 \leq \operatorname{Re}(z) \leq 0\} \\ &\cup \{|z| > 1/(2\sqrt{3}), 0 < \operatorname{Re}(z) \leq 1/4\} \cup \{|z - 1/3| > 1/6, 1/4 < \operatorname{Re}(z) \leq 1/2\}, \end{aligned}$$

where $W_{12} : e^{i\theta}/(2\sqrt{3}) \rightarrow e^{i(\pi-\theta)}/(2\sqrt{3})$ and $W_{12,4} : e^{i\theta}/6 + 1/3 \rightarrow e^{i(\pi-\theta)}/6 - 1/3$. Then, we have

$$(246) \quad \Gamma_0(12)+ = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{12}, W_{12,4} \rangle.$$

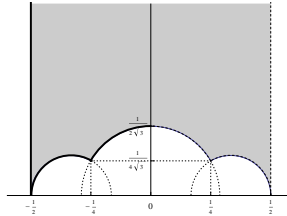


FIGURE 40. $\Gamma_0(12)+$

Valence formula. The cusps of $\Gamma_0(12)+$ are ∞ and 0, and the elliptic points are $i/(2\sqrt{3})$ and $\rho_{12,1} = -1/4 + i/(4\sqrt{3})$. Let f be a modular function of weight k for $\Gamma_0(12)+$, which is not identically zero. We have

$$(247) \quad v_\infty(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/(2\sqrt{3})}(f) + \frac{1}{2}v_{\rho_{12,1}}(f) + \sum_{\substack{p \in \Gamma_0(12)+ \setminus \mathbb{H} \\ p \neq i/(2\sqrt{3}), \rho_{12,1}}} v_p(f) = \frac{k}{2}.$$

Furthermore, the stabilizer of the elliptic point $i/(2\sqrt{3})$ (resp. $\rho_{12,1}$) is $\{\pm I, \pm W_{12}\}$ (resp. $\{\pm I, \pm W_{12,3}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(12)+$ of weight $k \geq 4$:

$$(248) \quad E_{k,12+}^\infty(z) := \frac{2^k 3^{k/2} E_k(12z) - 2 \cdot 3^{k/2} E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - 2 E_k(2z) + E_k(z)}{(3^{k/2} + 1)(2^k - 1)}.$$

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} 3n+1 & 3n/2 \\ -6n & -3n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{12+,6}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(12)+$ of weight $k \geq 4$:

$$(249) \quad E_{k,12+}^{-1/2}(z) := \frac{-2 \cdot 2^{k/2} (2^k 3^{k/2} E_k(12z) - 3^{k/2} (2^k + 1) E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - (2^k + 1) E_k(2z) + E_k(z))}{(3^{k/2} + 1)(2^k - 1)}.$$

We also have $\gamma_{-1/2}^{-1} \Gamma_0(12) + \gamma_{-1/2} = \Gamma_0(12)+$.

The space of modular forms. We define

$$\Delta_{12+}^\infty := \Delta_{12}^\infty \Delta_{12}^0 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4}, \quad \Delta_{12+}^{-1/2} := (\Delta_{12}^{-1/2})^2 (\Delta_{12}^{-1/6})^2,$$

which are 2nd semimodular forms for $\Gamma_0(12)+$ of weight 2.

Now, we have $M_k(\Gamma_0(12)+) = \mathbb{C} E_{k,12+}^\infty \oplus \mathbb{C} E_{k,12+}^{-1/2} \oplus S_k(\Gamma_0(12)+)$ and $S_k(\Gamma_0(12)+) = \Delta_{12+} M_{k-4}(\Gamma_0(12)+)$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(12)+) = E_{2,12+}' M_{4n}(\Gamma_0(12)+)$ and

$$\begin{aligned} M_{4n}(\Gamma_0(12)+) &= \mathbb{C} (E_{4,12+}^\infty)^n \oplus \mathbb{C} (E_{4,12+}^\infty)^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C} E_{4,12+}^\infty (\Delta_{12+})^{n-1} \\ &\quad \oplus \mathbb{C} (E_{4,12+}^{-1/2})^n \oplus \mathbb{C} (E_{4,12+}^{-1/2})^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C} E_{4,12+}^{-1/2} (\Delta_{12+})^{n-1} \oplus \mathbb{C} (\Delta_{12+})^n. \end{aligned}$$

Furthermore, we can write

$$M_{4n}(\Gamma_0(12)+) = \mathbb{C} (\Delta_{12+}^\infty)^{2n} \oplus \mathbb{C} (\Delta_{12+}^\infty)^{2n-1} \Delta_{12+}^{-1/2} \oplus \cdots \oplus \mathbb{C} (\Delta_{12+}^{-1/2})^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(12)+$:

$$(250) \quad \begin{aligned} J_{12+} &:= \Delta_{12+}^{-1/2} / \Delta_{12+}^\infty (= \eta^{-6}(z) \eta^{12}(2z) \eta^{-6}(3z) \eta^{-6}(4z) \eta^{12}(6z) \eta^{-6}(12z)) \\ &= \frac{1}{q} + 6 + 15q + 32q^2 + 87q^3 + \cdots, \end{aligned}$$

where $v_\infty(J_{12+}) = -1$ and $v_{-1/2}(J_{12+}) = 1$. Then, we have

$$(251) \quad J_{12+} : \partial \mathbb{F}_{12+} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [0, 16] \subset \mathbb{R}.$$

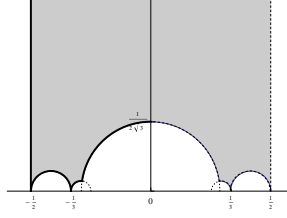
12.2. $\Gamma_0(12) + 12 = \Gamma_0^*(12)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0^*(12)$ as follows:

$$(252) \quad \begin{aligned} \mathbb{F}_{12+12} &= \{ |z + 5/12| \geq 1/12, -1/2 \leq \operatorname{Re}(z) < -1/3 \} \cup \{ |z + 7/24| \geq 1/24, -1/3 \leq \operatorname{Re}(z) < -2/7 \} \\ &\quad \cup \left\{ |z| \geq 1/(2\sqrt{3}), -2/7 \leq \operatorname{Re}(z) \leq 0 \right\} \cup \left\{ |z| > 1/(2\sqrt{3}), 0 < \operatorname{Re}(z) \leq 2/7 \right\} \\ &\quad \cup \{ |z - 7/24| > 1/24, 2/7 < \operatorname{Re}(z) \leq 1/3 \} \cup \{ |z - 5/12| > 1/12, 1/3 < \operatorname{Re}(z) < 1/2 \}, \end{aligned}$$

where $W_{12} : e^{i\theta}/(2\sqrt{3}) \rightarrow e^{i(\pi-\theta)}/(2\sqrt{3})$, $\begin{pmatrix} -7 & 2 \\ 24 & -7 \end{pmatrix} : (e^{i\theta} + 7)/24 \rightarrow (e^{i(\pi-\theta)} - 7)/24$, and $\begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(253) \quad \Gamma_0^*(12) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{12}, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ 24 & 7 \end{pmatrix} \rangle.$$

FIGURE 41. $\Gamma_0^*(12)$

Valence formula. The cusps of $\Gamma_0^*(12)$ are ∞ , $-1/3$, and $-1/2$, and the elliptic points are $i/(2\sqrt{3})$ and $\rho_{12,2} = -2/7 + i/(14\sqrt{3})$. Let f be a modular function of weight k for $\Gamma_0^*(12)$, which is not identically zero. We have

$$(254) \quad v_\infty(f) + v_{-1/3}(f) + v_{-1/2}(f) + \frac{1}{2}v_{i/(2\sqrt{3})}(f) + \frac{1}{2}v_{\rho_{12,2}}(f) + \sum_{\substack{p \in \Gamma_0^*(12) \setminus \mathbb{H} \\ p \neq i/(2\sqrt{3}), \rho_{12,2}}} v_p(f) = k.$$

Furthermore, the stabilizer of the elliptic point $i/(2\sqrt{3})$ (resp. $\rho_{12,2}$) is $\{\pm I, \pm W_{12}\}$ (resp. $\{\pm I, \pm \begin{pmatrix} 7 & -2 \\ -24 & 7 \end{pmatrix} W_{12}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0^*(12)$ of weight $k \geq 4$:

$$(255) \quad E_{k,12+12}^\infty(z) := \frac{2^k 3^k E_k(12z) + 3^{k/2}(3^{k/2} - 1)E_k(6z) - 2^k 3^{k/2} E_k(4z) - 3^k E_k(3z) - (3^{k/2} - 1)E_k(2z) + E_k(z)}{(3^k - 1)(2^k - 1)}.$$

For the cusp $-1/3$. We have $\Gamma_{-1/3} = \{\pm \begin{pmatrix} 12n+1 & 4n \\ -36n & -12n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/3} = W_{12,4}$, and we have the Eisenstein series for the cusp $-1/3$ associated with $\Gamma_0^*(12)$ of weight $k \geq 4$:

$$(256) \quad E_{k,12+12}^{-1/3}(z) := \frac{-(2^k 3^{k/2} E_k(12z) + 3^{k/2}(3^{k/2} - 1)E_k(6z) - 2^k 3^{k/2} E_k(4z) - 3^k E_k(3z) + (3^{k/2} - 1)E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)}.$$

We also have $\gamma_{-1/3}^{-1} \Gamma_0^*(12) \gamma_{-1/3} = \Gamma_0^*(12)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm \begin{pmatrix} 6n+1 & 3n \\ -12n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{12-,6}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0^*(12)$ of weight $k \geq 4$:

$$(257) \quad E_{k,12+12}^{-1/2}(z) := \frac{-(2^k 3^{k/2} E_k(12z) - 3^{k/2}(2^k + 1)E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - (2^k + 1)E_k(2z) + E_k(z))}{(3^{k/2} + 1)(2^k - 1)}.$$

Note that we have $E_{k,12+12}^{-1/2} = 2^{-1} 2^{-k/2} E_{k,12+}^{-1/2}$. Note that $\gamma_{-1/2}^{-1} \Gamma_0^*(12) \gamma_{-1/2} \neq \Gamma_0^*(12)$.

The space of modular forms. We define

$$\Delta_{12+12}^\infty := \Delta_{12}^\infty \Delta_{12}^0, \quad \Delta_{12+12}^{-1/3} := \Delta_{12}^{-1/3} \Delta_{12}^{-1/4}, \quad \Delta_{12+12}^{-1/2} := \Delta_{12}^{-1/2} \Delta_{12}^{-1/6},$$

which are 4th semimodular forms for $\Gamma_0^*(12)$ of weight 1. Furthermore, we define

$$\begin{aligned} \Delta_{12,1,12+12} &:= (E_{1,12+12})'^5 E_{2,12+} \Delta_{12+12}^{-1/3} \Delta_{12+12}^{-1/2} \Delta_{12}, & \Delta_{12,2,12+12} &:= (E_{1,12+12})'^5 E_{2,12+} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/2} \Delta_{12}, \\ \Delta_{12,3,12+12} &:= (E_{1,12+12})'^5 E_{2,12+} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/3} \Delta_{12}, & \Delta_{12,4,12+12} &:= (E_{1,12+12})'^4 \Delta_{12+12}^{-1/3} \Delta_{12+12}^{-1/2} (\Delta_{12})^2, \\ \Delta_{12,5,12+12} &:= (E_{1,12+12})'^4 \Delta_{12+12}^\infty \Delta_{12+12}^{-1/2} (\Delta_{12})^2, & \Delta_{12,6,12+12} &:= (E_{1,12+12})'^4 \Delta_{12+12}^\infty \Delta_{12+12}^{-1/3} (\Delta_{12})^2, \\ \Delta_{12,7,12+12} &:= (\Delta_{12+12}^{-1/3})^2 \Delta_{12+12}^{-1/2} (\Delta_{12})^3, & \Delta_{12,8,12+12} &:= \Delta_{12+12}^\infty (\Delta_{12+12}^{-1/2})^2 (\Delta_{12})^3, \\ \Delta_{12,9,12+12} &:= (\Delta_{12+12}^\infty)^2 \Delta_{12+12}^{-1/3} (\Delta_{12})^3. \end{aligned}$$

In addition, we denote

$$\begin{aligned}
A_1 &= E_{1,12+12}' \Delta_{12} (\mathbb{C} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/2} \oplus \mathbb{C} \Delta_{12+12}^{-1/3} \Delta_{12+12}^{-1/2} \oplus \mathbb{C} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/3}), \\
A_2 &= (\Delta_{12})^2 (\mathbb{C} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/2} \oplus \mathbb{C} \Delta_{12+12}^{-1/3} \Delta_{12+12}^{-1/2} \oplus \mathbb{C} \Delta_{12+12}^\infty \Delta_{12+12}^{-1/3}), \\
B_1 &= \mathbb{C} (E_{12,12+12}^\infty)^n \oplus \mathbb{C} (E_{12,12+12}^\infty)^{n-1} \Delta_{12,1,12+12} \oplus \mathbb{C} (E_{12,12+12}^\infty)^{n-1} \Delta_{12,4,12+12} \\
&\quad \oplus \mathbb{C} (E_{12,12+12}^\infty)^{n-1} \Delta_{12,7,12+12} \oplus \mathbb{C} (E_{12,12+12}^\infty)^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+12}^\infty)^{n-2} \Delta_{12,1,12+12} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,7,12+12} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
B_2 &= \mathbb{C} (E_{12,12+12}^{-1/3})^n \oplus \mathbb{C} (E_{12,12+12}^{-1/3})^{n-1} \Delta_{12,2,12+12} \oplus \mathbb{C} (E_{12,12+12}^{-1/3})^{n-1} \Delta_{12,5,12+12} \\
&\quad \oplus \mathbb{C} (E_{12,12+12}^{-1/3})^{n-1} \Delta_{12,8,12+12} \oplus \mathbb{C} (E_{12,12+12}^{-1/3})^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+12}^{-1/3})^{n-2} \Delta_{12,2,12+12} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,8,12+12} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
B_3 &= \mathbb{C} (E_{12,12+12}^{-1/2})^n \oplus \mathbb{C} (E_{12,12+12}^{-1/2})^{n-1} \Delta_{12,3,12+12} \oplus \mathbb{C} (E_{12,12+12}^{-1/2})^{n-1} \Delta_{12,6,12+12} \\
&\quad \oplus \mathbb{C} (E_{12,12+12}^{-1/2})^{n-1} \Delta_{12,9,12+12} \oplus \mathbb{C} (E_{12,12+12}^{-1/2})^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+12}^{-1/2})^{n-2} \Delta_{12,3,12+12} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,9,12+12} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}.
\end{aligned}$$

Now, we have

$$M_k(\Gamma_0^*(12)) = \mathbb{C} E_{k,12+12}^\infty \oplus \mathbb{C} E_{k,12+12}^{-1/3} \oplus \mathbb{C} E_{k,12+12}^{-1/2} \oplus S_k(\Gamma_0^*(12)),$$

$$S_k(\Gamma_0^*(12)) = (\mathbb{C} \Delta_{12,1,12+12} \oplus \mathbb{C} \Delta_{12,2,12+12} \oplus \cdots \oplus \mathbb{C} \Delta_{12,9,12+12} \oplus \mathbb{C} (\Delta_{12})^4) M_{k-12}(\Gamma_0^*(12))$$

for every even integer $k \geq 4$. Then, we have

$$\begin{aligned}
M_{12n}(\Gamma_0^*(12)) &= B_1 \oplus B_2 \oplus B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
M_{12n+2}(\Gamma_0^*(12)) &= E_{2,12+}' M_{12n}(\Gamma_0^*(12)) \oplus \mathbb{C} E_{1,12+12}' \Delta_{12+12}^\infty (\Delta_{12})^{4n}, \\
M_{12n+4}(\Gamma_0^*(12)) &= E_{4,12+12}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{4,12+12}^{-1/3} (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{4,12+12}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus (\mathbb{C} E_{1,12+12}' \oplus \mathbb{C} \Delta_{12+12}^\infty) (\Delta_{12})^{4n+1}, \\
M_{12n+6}(\Gamma_0^*(12)) &= E_{6,12+12}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{6,12+12}^{-1/3} (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{6,12+12}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus A_1 (\Delta_{12})^{4n}, \\
M_{12n+8}(\Gamma_0^*(12)) &= E_{8,12+12}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{8,12+12}^{-1/3} (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{8,12+12}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus E_{2,12+}' A_1 (\Delta_{12})^{4n} \oplus A_2 (\Delta_{12})^{4n}, \\
M_{12n+10}(\Gamma_0^*(12)) &= E_{10,12+12}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{10,12+12}^{-1/3} (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{10,12+12}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus \mathbb{C} (E_{1,12+12}')^4 A_1 (\Delta_{12})^{4n} \oplus E_{2,12+}' A_2 (\Delta_{12})^{4n+1} \oplus \mathbb{C} E_{1,12+12}' (\Delta_{12})^{4n+3}.
\end{aligned}$$

Furthermore, we can write

$$M_k(\Gamma_0^*(12)) = E_{k,12+12}' (\mathbb{C} (\Delta_{12+12}^\infty)^n \oplus \mathbb{C} (\Delta_{12+12}^\infty)^{n-1} \Delta_{12+12}^{-1/3} \oplus \cdots \oplus \mathbb{C} (\Delta_{12+12}^{-1/3})^n),$$

where $n = \dim(M_k(\Gamma_0^*(12))) - 1 = [k - 2(k/4 - \lfloor k/4 \rfloor)]$, and where $E_{k,12+12}' := 1$ and $E_{1,12+12}'$, when $k \equiv 0$ and $2 \pmod{4}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0^*(12)$:

$$(258) \quad J_{12+12} := \Delta_{12+12}^{-1/3} / \Delta_{12+12}^\infty (= \eta^{-4}(z) \eta^4(3z) \eta^4(4z) \eta^{-4}(12z)) = \frac{1}{q} + 4 + 14q + 36q^2 + 85q^3 + \cdots,$$

where $v_\infty(J_{12+12}) = -1$ and $v_{-1/3}(J_{12+12}) = 1$. Then, we have

$$(259) \quad J_{12+12} : \partial \mathbb{F}_{12+12} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-1, 7 + 4\sqrt{3}] \subset \mathbb{R}.$$

12.3. $\Gamma_0(12) + 4$.

We have $\Gamma_0(12) + 4 = T_{1/2}^{-1}\Gamma_0(6)T_{1/2}$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(12) + 4$ as follows:

$$(260) \quad \mathbb{F}_{12+4} = \{|z + 1/3| \geq 1/6, -1/2 \leq \operatorname{Re}(z) < -1/6\} \cup \{|z + 1/12| \geq 1/12, -1/6 \leq \operatorname{Re}(z) \leq 0\} \\ \cup \{|z - 1/12| > 1/12, 0 < \operatorname{Re}(z) \leq 1/6\} \cup \{|z - 1/3| > 1/6, 1/6 < \operatorname{Re}(z) < 1/2\},$$

where $\begin{pmatrix} -1 & 0 \\ 12 & -1 \end{pmatrix} : (e^{i\theta} + 1)/12 \rightarrow (e^{i(\pi-\theta)} - 1)/12$ and $W_{12,4} : 1/3 + e^{i\theta}/6 \rightarrow -1/3 + e^{i(\pi-\theta)}/6$. Then, we have

$$(261) \quad \Gamma_0(12) + 4 = \langle -I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, W_{12,4}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \rangle.$$

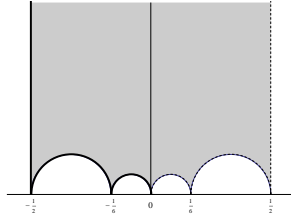


FIGURE 42. $\Gamma_0(12) + 4$

Valence formula. The cusps of $\Gamma_0(12) + 4$ are ∞ , 0 , $-1/2$, and $-1/6$. Let f be a modular function of weight k for $\Gamma_0(12) + 4$, which is not identically zero. We have

$$(262) \quad v_\infty(f) + v_0(f) + v_{-1/2}(f) + v_{-1/6}(f) + \sum_{p \in \Gamma_0(12)+4 \setminus \mathbb{H}} v_p(f) = k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(12) + 4$:

$$(263) \quad E_{k,12+4}^\infty(z) := \frac{2^k 3^k E_k(12z) - 2 \cdot 3^k E_k(6z) - 2^k E_k(4z) + 3^k E_k(3z) + E_k(2z) - E_k(z)}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

For the cusp 0 . We have $\Gamma_0 = \{\pm \begin{pmatrix} 1 & 0 \\ 12n & 1 \end{pmatrix} ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{12}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(12) + 4$:

$$(264) \quad E_{k,12+4}^0(z) := \frac{3^{k/2}(2^k E_k(12z) - 2E_k(6z) - 2^k E_k(4z) + E_k(3z) + 2E_k(2z) - E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

We also have $\gamma_0^{-1}(\Gamma_0(12) + 4)\gamma_0 = \Gamma_0(12) + 4$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} 3n+1 & 3n/2 \\ -6n & -3n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{12+6}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(12) + 4$ of weight $k \geq 4$:

$$(265) \quad E_{k,12+4}^{-1/2}(z) := \frac{2 \cdot 2^{k/2} 3^{k/2} (2^k 3^k E_k(12z) - 3^k (2^k + 1) E_k(6z) - 2^k E_k(4z) + 3^k E_k(3z) + (2^k + 1) E_k(2z) - E_k(z))}{(3^k - 1)(2^k - 1)}.$$

We also have $\gamma_{-1/2}^{-1}(\Gamma_0(12) + 4)\gamma_{-1/2} = \Gamma_0(12) + 4$.

For the cusp $-1/6$. We have $\Gamma_{-1/6} = \left\{ \pm \begin{pmatrix} 3n+1 & n/2 \\ -18n & -3n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/6} = W_{12+,2}$, and we have the Eisenstein series for the cusp $-1/6$ associated with $\Gamma_0(12) + 4$ of weight $k \geq 4$:

$$(266) \quad E_{k,12+4}^{-1/6}(z) := \frac{-2 \cdot 2^{k/2} (2^k 3^{k/2} E_k(12z) - 3^{k/2} (2^k + 1) E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - (2^k + 1) E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)}.$$

We also have $\gamma_{-1/6}^{-1}(\Gamma_0(12) + 4) \gamma_{-1/6} = \Gamma_0(12) + 4$.

The space of modular forms. We define

$$\begin{aligned} \Delta_{12+4}^\infty &:= \Delta_{12}^\infty \Delta_{12}^{-1/3}, & \Delta_{12+4}^0 &:= \Delta_{12}^0 \Delta_{12}^{-1/4}, \\ \Delta_{12+4}^{-1/2} &:= (\Delta_{12}^{-1/2})^2, & \Delta_{12+4}^{-1/6} &:= (\Delta_{12}^{-1/6})^2, \end{aligned}$$

which are 4th semimodular forms for $\Gamma_0^*(12)$ of weight 1.

We have $M_k(\Gamma_0(12) + 4) = \mathbb{C}E_{k,12+4}^\infty \oplus \mathbb{C}E_{k,12+4}^0 \oplus \mathbb{C}E_{k,12+4}^{-1/2} \oplus \mathbb{C}E_{k,12+4}^{-1/6} \oplus S_k(\Gamma_0(12) + 4)$ and $S_k(\Gamma_0(12) + 4) = \Delta_{12+} M_{k-4}(\Gamma_0(12) + 4)$ for every even integer $k \geq 4$. Then, we have $M_{4n+2}(\Gamma_0(12) + 4) = E_{2,12+} M_{4n}(\Gamma_0(12) + 4) \oplus \mathbb{C}(E_{1,12+12}')^2(\Delta_{12+})^n \oplus \mathbb{C}(E_{1,12+3}')^2(\Delta_{12+})^n$ and

$$\begin{aligned} M_{4n}(\Gamma_0(12) + 4) &= \mathbb{C}(E_{4,12+4}^\infty)^n \oplus \mathbb{C}(E_{4,12+4}^\infty)^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C}E_{4,12+4}^\infty (\Delta_{12+})^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,12+4}^0)^n \oplus \mathbb{C}(E_{4,12+4}^0)^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C}E_{4,12+4}^0 (\Delta_{12+})^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,12+4}^{-1/2})^n \oplus \mathbb{C}(E_{4,12+4}^{-1/2})^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C}E_{4,12+4}^{-1/2} (\Delta_{12+})^{n-1} \\ &\quad \oplus \mathbb{C}(E_{4,12+4}^{-1/6})^n \oplus \mathbb{C}(E_{4,12+4}^{-1/6})^{n-1} \Delta_{12+} \oplus \cdots \oplus \mathbb{C}E_{4,12+4}^{-1/6} (\Delta_{12+})^{n-1} \oplus \mathbb{C}(\Delta_{12+})^n. \end{aligned}$$

Furthermore, we can write

$$M_{2n}(\Gamma_0(12) + 4) = \mathbb{C}(\Delta_{12+4}^\infty)^{2n} \oplus \mathbb{C}(\Delta_{12+4}^\infty)^{2n-1} \Delta_{12+4}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{12+4}^0)^{2n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(12) + 4$:

$$(267) \quad \begin{aligned} J_{12+4} &:= \Delta_{12+4}^0 / \Delta_{12+4}^\infty (= \eta^4(z) \eta^{-4}(2z) \eta^{-4}(3z) \eta^4(4z) \eta^4(6z) \eta^{-4}(12z)) \\ &= \frac{1}{q} - 4 + 6q - 4q^2 - 3q^3 + \cdots, \end{aligned}$$

where $v_\infty(J_{12+4}) = -1$ and $v_0(J_{12+4}) = 1$. Then, we have

$$(268) \quad J_{12+4} : \partial \mathbb{F}_{12+4} \setminus \{z \in \mathbb{H} ; Re(z) = \pm 1/2\} \rightarrow [-9, 0] \subset \mathbb{R}.$$

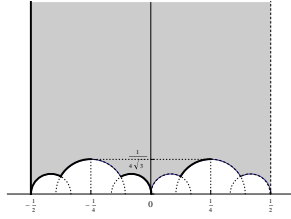
12.4. $\Gamma_0(12) + 3$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(12) + 3$ as follows:

$$(269) \quad \begin{aligned} \mathbb{F}_{12+3} &= \{ |z + 5/12| \geq 1/12, -1/2 \leq Re(z) \leq -3/8 \} \cup \{ |z + 1/4| \geq 1/(4\sqrt{3}), -3/8 < Re(z) \leq -1/4 \} \\ &\quad \cup \{ |z + 1/4| > 1/(4\sqrt{3}), -1/4 < Re(z) < -1/8 \} \cup \{ |z + 1/12| \geq 1/12, -1/8 \leq Re(z) \leq 0 \} \\ &\quad \cup \{ |z - 1/12| > 1/12, 0 < Re(z) < 1/8 \} \cup \{ |z - 1/4| \geq 1/(4\sqrt{3}), 1/8 \leq Re(z) \leq 1/4 \} \\ &\quad \cup \{ |z - 1/4| > 1/(4\sqrt{3}), 1/4 < Re(z) \leq 3/8 \} \cup \{ |z - 5/12| > 1/12, 3/8 < Re(z) < 1/2 \}, \end{aligned}$$

where $W_{12,3} : -1/4 + e^{i\theta}/(4\sqrt{3}) \rightarrow -1/4 + e^{i(\pi-\theta)}/(4\sqrt{3})$, $(\frac{7}{24} \frac{2}{7}) W_{12,3} : 1/4 + e^{i\theta}/(4\sqrt{3}) \rightarrow 1/4 + e^{i(\pi-\theta)}/(4\sqrt{3})$, $(\frac{-1}{12} \frac{0}{-1}) : (e^{i\theta} + 1)/12 \rightarrow (e^{i(\pi-\theta)} - 1)/12$, and $(\frac{-5}{12} \frac{2}{-5}) : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(270) \quad \Gamma_0(12) + 3 = \langle (\frac{1}{0} \frac{1}{1}), W_{12,3}, (\frac{1}{12} \frac{0}{1}), (\frac{5}{12} \frac{2}{5}) \rangle.$$

FIGURE 43. $\Gamma_0(12) + 3$

Valence formula. The cusps of $\Gamma_0(12) + 3$ are ∞ , 0, and $-1/2$, and the elliptic points are $\rho_{12,1}$ and $\rho_{12,3} = 1/4 + i/(4\sqrt{3})$. Let f be a modular function of weight k for $\Gamma_0(12) + 3$, which is not identically zero. We have

$$(271) \quad v_\infty(f) + v_{-1/3}(f) + v_{-1/2}(f) + \frac{1}{2}v_{\rho_{12,1}}(f) + \frac{1}{2}v_{\rho_{12,3}}(f) + \sum_{\substack{p \in \Gamma_0(12)+3 \backslash \mathbb{H} \\ p \neq \rho_{12,1}, \rho_{12,3}}} v_p(f) = k.$$

Furthermore, the stabilizer of the elliptic point $\rho_{12,1}$ (resp. $\rho_{12,3}$) is $\{\pm I, \pm W_{12,3}\}$ (resp. $\{\pm I, \pm (\frac{7}{24} \frac{2}{7}) W_{12,3}\}$).

For the cusp ∞ . We have $\Gamma_\infty = \{\pm (\frac{1}{0} \frac{n}{1}) ; n \in \mathbb{Z}\}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(12) + 3$:

$$(272) \quad E_{k,12+3}^\infty(z) := \frac{2^k 3^{k/2} E_k(12z) - 3^{k/2} E_k(6z) + 2^k E_k(4z) - E_k(2z)}{(3^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12+3}^\infty(z) = E_{k,6+3}^\infty(2z)$.

For the cusp 0. We have $\Gamma_0 = \{\pm (\frac{1}{12n} \frac{0}{1}) ; n \in \mathbb{Z}\}$ and $\gamma_0 = W_{12}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(12) + 3$:

$$(273) \quad E_{k,12+3}^0(z) := \frac{-(3^{k/2} E_k(6z) - 3^{k/2} E_k(3z) + E_k(2z) - E_k(z))}{(3^{k/2} + 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12+3}^0(z) = 2^{-k/2} E_{k,6+3}^0(z)$. We also have $\gamma_0^{-1}(\Gamma_0(12) + 3)\gamma_0 = \Gamma_0(12) + 3$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \{\pm (\frac{6n+1}{-12n} \frac{3n}{-6n+1}) ; n \in \mathbb{Z}\}$ and $\gamma_{-1/2} = W_{12-,6}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(12) + 3$ of weight $k \geq 4$:

$$(274) \quad E_{k,12+3}^{-1/2}(z) := \frac{-(2^k 3^{k/2} E_k(12z) - 3^{k/2} (2^k + 1) E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - (2^k + 1) E_k(2z) + E_k(z))}{(3^{k/2} + 1)(2^k - 1)}.$$

Note that we have $E_{k,12+3}^{-1/2}(z) = E_{k,12+12}^{-1/2}(z) = 2^{-1} 2^{-k/2} E_{k,12+}^{-1/2}(z)$. We also have $\gamma_{-1/2}^{-1}(\Gamma_0(12) + 3)\gamma_{-1/2} = \Gamma_0(12) + 3$.

The space of modular forms. We define

$$\Delta_{12+3}^\infty := \Delta_{12}^\infty \Delta_{12}^{-1/4}, \quad \Delta_{12+3}^0 := \Delta_{12}^0 \Delta_{12}^{-1/3}, \quad \Delta_{12+3}^{-1/2} := \Delta_{12}^{-1/2} \Delta_{12}^{-1/6},$$

which are 4th semimodular forms for $\Gamma_0(12) + 3$ of weight 1. Furthermore, we define

$$\begin{aligned} \Delta_{12,1,12+3} &:= (E_{1,12+3}')^5 E_{2,12+} {}' \Delta_{12+3}^0 \Delta_{12+3}^{-1/2} \Delta_{12}, & \Delta_{12,2,12+3} &:= (E_{1,12+3}')^5 E_{2,12+} {}' \Delta_{12+3}^\infty \Delta_{12+3}^{-1/2} \Delta_{12}, \\ \Delta_{12,3,12+3} &:= (E_{1,12+3}')^5 E_{2,12+} {}' \Delta_{12+3}^\infty \Delta_{12+3}^0 \Delta_{12}, & \Delta_{12,4,12+3} &:= (E_{1,12+3}')^4 \Delta_{12+3}^0 \Delta_{12+3}^{-1/2} (\Delta_{12})^2, \\ \Delta_{12,5,12+3} &:= (E_{1,12+3}')^4 \Delta_{12+3}^\infty \Delta_{12+3}^{-1/2} (\Delta_{12})^2, & \Delta_{12,6,12+3} &:= (E_{1,12+3}')^4 \Delta_{12+3}^\infty \Delta_{12+3}^0 (\Delta_{12})^2, \\ \Delta_{12,7,12+3} &:= (\Delta_{12+3}^0)^2 \Delta_{12+3}^{-1/2} (\Delta_{12})^3, & \Delta_{12,8,12+3} &:= \Delta_{12+3}^\infty (\Delta_{12+3}^{-1/2})^2 (\Delta_{12})^3, \\ \Delta_{12,9,12+3} &:= (\Delta_{12+3}^\infty)^2 \Delta_{12+3}^0 (\Delta_{12})^3. \end{aligned}$$

In addition, we denote

$$\begin{aligned}
A_1 &= E_{1,12+3}' \Delta_{12} (\mathbb{C} \Delta_{12+3}^\infty \Delta_{12+3}^{-1/2} \oplus \mathbb{C} \Delta_{12+3}^0 \Delta_{12+3}^{-1/2} \oplus \mathbb{C} \Delta_{12+3}^\infty \Delta_{12+3}^0), \\
A_2 &= (\Delta_{12})^2 (\mathbb{C} \Delta_{12+3}^\infty \Delta_{12+3}^{-1/2} \oplus \mathbb{C} \Delta_{12+3}^0 \Delta_{12+3}^{-1/2} \oplus \mathbb{C} \Delta_{12+3}^\infty \Delta_{12+3}^0), \\
B_1 &= \mathbb{C} (E_{12,12+3}^\infty)^n \oplus \mathbb{C} (E_{12,12+3}^\infty)^{n-1} \Delta_{12,1,12+3} \oplus \mathbb{C} (E_{12,12+3}^\infty)^{n-1} \Delta_{12,4,12+3} \\
&\quad \oplus \mathbb{C} (E_{12,12+3}^\infty)^{n-1} \Delta_{12,7,12+3} \oplus \mathbb{C} (E_{12,12+3}^\infty)^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+3}^\infty)^{n-2} \Delta_{12,1,12+3} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,7,12+3} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
B_2 &= \mathbb{C} (E_{12,12+3}^0)^n \oplus \mathbb{C} (E_{12,12+3}^0)^{n-1} \Delta_{12,2,12+3} \oplus \mathbb{C} (E_{12,12+3}^0)^{n-1} \Delta_{12,5,12+3} \\
&\quad \oplus \mathbb{C} (E_{12,12+3}^0)^{n-1} \Delta_{12,8,12+3} \oplus \mathbb{C} (E_{12,12+3}^0)^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+3}^0)^{n-2} \Delta_{12,2,12+3} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,8,12+3} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
B_3 &= \mathbb{C} (E_{12,12+3}^{-1/2})^n \oplus \mathbb{C} (E_{12,12+3}^{-1/2})^{n-1} \Delta_{12,3,12+3} \oplus \mathbb{C} (E_{12,12+3}^{-1/2})^{n-1} \Delta_{12,6,12+3} \\
&\quad \oplus \mathbb{C} (E_{12,12+3}^{-1/2})^{n-1} \Delta_{12,9,12+3} \oplus \mathbb{C} (E_{12,12+3}^{-1/2})^{n-1} (\Delta_{12})^4 \oplus \mathbb{C} (E_{12,12+3}^{-1/2})^{n-2} \Delta_{12,3,12+3} (\Delta_{12})^4 \\
&\quad \oplus \cdots \oplus \mathbb{C} \Delta_{12,9,12+3} (\Delta_{12})^{4(n-1)} \oplus \mathbb{C} (\Delta_{12})^{4n}.
\end{aligned}$$

Now, we have

$$M_k(\Gamma_0(12) + 3) = \mathbb{C} E_{k,12+3}^\infty \oplus \mathbb{C} E_{k,12+3}^0 \oplus \mathbb{C} E_{k,12+3}^{-1/2} \oplus S_k(\Gamma_0(12) + 3),$$

$$S_k(\Gamma_0(12) + 3) = (\mathbb{C} \Delta_{12,1,12+3} \oplus \mathbb{C} \Delta_{12,2,12+3} \oplus \cdots \oplus \mathbb{C} \Delta_{12,9,12+3} \oplus \mathbb{C} (\Delta_{12})^4) M_{k-12}(\Gamma_0(12) + 3)$$

for every even integer $k \geq 4$. Then, we have

$$\begin{aligned}
M_{12n}(\Gamma_0(12) + 3) &= B_1 \oplus B_2 \oplus B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}, \\
M_{12n+2}(\Gamma_0(12) + 3) &= E_{2,12+3}' M_{12n}(\Gamma_0(12) + 3) \oplus \mathbb{C} E_{1,12+3}' \Delta_{12+3}^\infty (\Delta_{12})^{4n}, \\
M_{12n+4}(\Gamma_0(12) + 3) &= E_{4,12+3}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{4,12+3}^0 (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{4,12+3}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus (\mathbb{C} E_{1,12+3}' \oplus \mathbb{C} \Delta_{12+3}^\infty) (\Delta_{12})^{4n+1}, \\
M_{12n+6}(\Gamma_0(12) + 3) &= E_{6,12+3}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{6,12+3}^0 (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{6,12+3}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus A_1 (\Delta_{12})^{4n}, \\
M_{12n+8}(\Gamma_0(12) + 3) &= E_{8,12+3}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{8,12+3}^0 (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{8,12+3}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus E_{2,12+3}' A_1 (\Delta_{12})^{4n} \oplus A_2 (\Delta_{12})^{4n}, \\
M_{12n+10}(\Gamma_0(12) + 3) &= E_{10,12+3}^\infty (B_1 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{10,12+3}^0 (B_2 \oplus \mathbb{C} (\Delta_{12})^{4n}) \oplus E_{10,12+3}^{-1/2} (B_3 \oplus \mathbb{C} (\Delta_{12})^{4n}) \\
&\quad \oplus \mathbb{C} (E_{1,12+3}')^4 A_1 (\Delta_{12})^{4n} \oplus E_{2,12+3}' A_2 (\Delta_{12})^{4n+1} \oplus \mathbb{C} E_{1,12+3}' (\Delta_{12})^{4n+3}.
\end{aligned}$$

Furthermore, we can write

$$M_k(\Gamma_0(12) + 3) = E_{\bar{k},12+3}' (\mathbb{C} (\Delta_{12+3}^\infty)^n \oplus \mathbb{C} (\Delta_{12+3}^\infty)^{n-1} \Delta_{12+3}^0 \oplus \cdots \oplus \mathbb{C} (\Delta_{12+3}^0)^n),$$

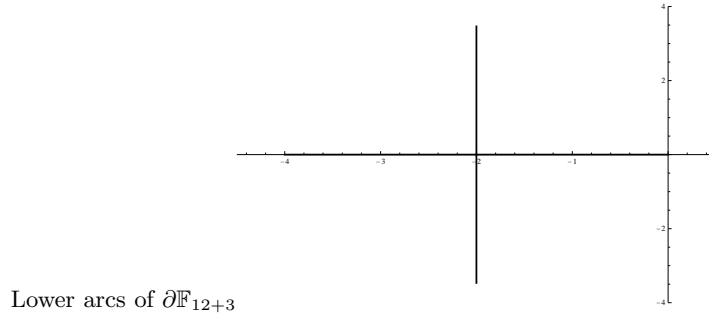
where $n = \dim(M_k(\Gamma_0(12) + 3)) - 1 = \lfloor k - 2(k/4 - \lfloor k/4 \rfloor) \rfloor$, and where $E_{\bar{k},12+3}' := 1$ and $E_{1,12+3}'$, when $k \equiv 0$ and $2 \pmod{4}$, respectively.

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(12) + 3$:

$$(275) \quad J_{12+3} := \Delta_{12+3}^0 / \Delta_{12+3}^\infty (= \eta^2(z) \eta^2(3z) \eta^{-2}(4z) \eta^{-2}(12z)) = \frac{1}{q} - 2 - q + 7q^3 - 9q^5 + \cdots,$$

where $v_\infty(J_{12+3}) = -1$ and $v_0(J_{12+3}) = 1$. Then, we have

$$\begin{aligned}
(276) \quad J_{12+3} : \quad &\{ |z + 1/12| = 1/12, -1/8 \leq \operatorname{Re}(z) \leq 0 \} \rightarrow -2 + [0, 2] \subset \mathbb{R}, \\
&\{ |z + 5/12| = 1/12, -1/2 \leq \operatorname{Re}(z) \leq -3/8 \} \rightarrow -2 - [0, 2] \subset \mathbb{R}, \\
&\{ |z + 1/4| = 1/(4\sqrt{3}), -3/8 \leq \operatorname{Re}(z) \leq -1/4 \} \rightarrow -2 + 2\sqrt{3}[0, 1], \\
&\{ |z - 1/4| = 1/(4\sqrt{3}), 1/8 \leq \operatorname{Re}(z) \leq 1/4 \} \rightarrow -2 - 2\sqrt{3}[0, 1].
\end{aligned}$$

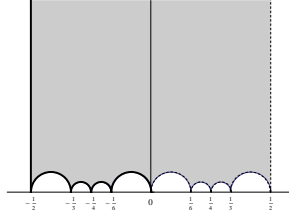
FIGURE 44. Image by J_{12+3} 12.5. $\Gamma_0(12)$.

Fundamental domain. We have a fundamental domain for $\Gamma_0(12)$ as follows:

$$(277) \quad \begin{aligned} \mathbb{F}_{12} = & \{ |z + 5/12| \geq 1/12, -1/2 \leq \operatorname{Re}(z) < -1/3 \} \cup \{ |z + 7/24| \geq 1/24, -1/3 \leq \operatorname{Re}(z) < -1/4 \} \\ & \cup \{ |z + 5/24| \geq 1/24, -1/4 \leq \operatorname{Re}(z) < -1/6 \} \cup \{ |z + 1/12| \geq 1/12, -1/6 \leq \operatorname{Re}(z) \leq 0 \} \\ & \cup \{ |z - 1/12| > 1/12, 0 < \operatorname{Re}(z) \leq 1/6 \} \cup \{ |z - 5/24| > 1/24, 1/6 < \operatorname{Re}(z) \leq 1/4 \} \\ & \cup \{ |z - 7/24| > 1/24, 1/4 < \operatorname{Re}(z) \leq 1/3 \} \cup \{ |z - 5/12| > 1/12, 1/3 < \operatorname{Re}(z) < 1/2 \}, \end{aligned}$$

where $\begin{pmatrix} -1 & 0 \\ 12 & -1 \end{pmatrix} : (e^{i\theta} + 1)/12 \rightarrow (e^{i(\pi-\theta)} - 1)/12$, $\begin{pmatrix} -5 & 1 \\ 24 & -5 \end{pmatrix} : (e^{i\theta} + 5)/24 \rightarrow (e^{i(\pi-\theta)} - 5)/24$, $\begin{pmatrix} -7 & 2 \\ 24 & -7 \end{pmatrix} : (e^{i\theta} + 7)/24 \rightarrow (e^{i(\pi-\theta)} - 7)/24$, and $\begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} : (e^{i\theta} + 5)/12 \rightarrow (e^{i(\pi-\theta)} - 5)/12$. Then, we have

$$(278) \quad \Gamma_0(12) = \langle -I, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 24 & 5 \end{pmatrix}, \begin{pmatrix} 7 & 2 \\ 24 & 7 \end{pmatrix} \rangle.$$

FIGURE 45. $\Gamma_0(12)$

Valence formula. The cusps of $\Gamma_0(12)$ are ∞ , 0 , $-1/3$, $-1/4$, $-1/2$, and $-1/6$. Let f be a modular function of weight k for $\Gamma_0(12)$, which is not identically zero. We have

$$(279) \quad v_\infty(f) + v_0(f) + v_{-1/3}(f) + v_{-1/4}(f) + v_{-1/2}(f) + v_{-1/6}(f) + \sum_{p \in \Gamma_0(12) \backslash \mathbb{H}} v_p(f) = 2k.$$

For the cusp ∞ . We have $\Gamma_\infty = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z} \}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(12)$:

$$(280) \quad E_{k,12}^\infty(z) := \frac{2^k 3^k E_k(12z) - 3^k E_k(6z) - 2^k E_k(4z) + E_k(2z)}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12}^\infty(z) = E_{k,6}^\infty(2z)$.

For the cusp 0 . We have $\Gamma_0 = \{ \pm \begin{pmatrix} 1 & 0 \\ 12n & 1 \end{pmatrix} ; n \in \mathbb{Z} \}$ and $\gamma_0 = W_{12}$, and we have the Eisenstein series for the cusp 0 associated with $\Gamma_0(12)$:

$$(281) \quad E_{k,12}^0(z) := \frac{3^{k/2}(E_k(6z) - E_k(3z) - E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12}^0(z) = 2^{-k/2} E_{k,6}^0(z)$. We also have $\gamma_0^{-1} \Gamma_0(12) \gamma_0 = \Gamma_0(12)$.

For the cusp $-1/3$. We have $\Gamma_{-1/3} = \left\{ \pm \begin{pmatrix} 12n+1 & 4n \\ -36n & -12n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/3} = W_{12,4}$, and we have the Eisenstein series for the cusp ∞ associated with $\Gamma_0(12)$:

$$(282) \quad E_{k,12}^{-1/3}(z) := \frac{-(3^k E_k(6z) - 3^k E_k(3z) - E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12}^{-1/3}(z) = 2^{-k/2} E_{k,6}^{-1/3}(z)$. We also have $\gamma_{-1/3}^{-1} \Gamma_0(12) \gamma_{-1/3} = \Gamma_0(12)$.

For the cusp $-1/4$. We have $\Gamma_{-1/4} = \left\{ \pm \begin{pmatrix} 12n+1 & 3n \\ -48n & -12n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/4} = W_{12,3}$, and we have the Eisenstein series for the cusp $-1/4$ associated with $\Gamma_0(12)$:

$$(283) \quad E_{k,12}^{-1/4}(z) := \frac{-3^{k/2}(2^k E_k(12z) - E_k(6z) - 2^k E_k(4z) + E_k(2z))}{(3^k - 1)(2^k - 1)} \quad \text{for } k \geq 4.$$

Note that we have $E_{k,12}^{-1/4}(z) = E_{k,6}^{-1/4}(2z)$. We also have $\gamma_{-1/4}^{-1} \Gamma_0(12) \gamma_{-1/4} = \Gamma_0(12)$.

For the cusp $-1/2$. We have $\Gamma_{-1/2} = \left\{ \pm \begin{pmatrix} 6n+1 & 3n \\ -12n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/2} = W_{12-,6}$, and we have the Eisenstein series for the cusp $-1/2$ associated with $\Gamma_0(12)$ of weight $k \geq 4$:

$$(284) \quad E_{k,12}^{-1/2}(z) := \frac{3^{k/2}(2^k 3^k E_k(12z) - 3^k(2^k + 1)E_k(6z) - 2^k E_k(4z) + 3^k E_k(3z) + (2^k + 1)E_k(2z) - E_k(z))}{(3^k - 1)(2^k - 1)}.$$

Note that we have $E_{k,12}^{-1/2}(z) = 2^{-1} 2^{-k/2} E_{k,12+4}^{-1/2}(z)$. We also have $\gamma_{-1/2}^{-1} \Gamma_0(12) \gamma_{-1/2} = \Gamma_0(12)$.

For the cusp $-1/6$. We have $\Gamma_{-1/6} = \left\{ \pm \begin{pmatrix} 6n+1 & n \\ -36n & -6n+1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$ and $\gamma_{-1/6} = W_{12-,2}$, and we have the Eisenstein series for the cusp $-1/6$ associated with $\Gamma_0(12)$ of weight $k \geq 4$:

$$(285) \quad E_{k,12}^{-1/6}(z) := \frac{-(2^k 3^{k/2} E_k(12z) - 3^{k/2}(2^k + 1)E_k(6z) + 2^k E_k(4z) + 3^{k/2} E_k(3z) - (2^k + 1)E_k(2z) + E_k(z))}{(3^k - 1)(2^k - 1)}.$$

Note that we have $E_{k,12}^{-1/6}(z) = 2^{-1} 2^{-k/2} E_{k,12+4}^{-1/6}(z)$. We also have $\gamma_{-1/6}^{-1} \Gamma_0(12) \gamma_{-1/6} = \Gamma_0(12)$.

The space of modular forms. We define

$$\begin{aligned} \Delta_{6,1,12-} &:= (\Delta_{12}^0)^2 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} \Delta_{12}^{-1/2} \Delta_{12}^{-1/6}, & \Delta_{6,2,12-} &:= (\Delta_{12}^\infty)^2 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} \Delta_{12}^{-1/2} \Delta_{12}^{-1/6}, \\ \Delta_{6,3,12-} &:= \Delta_{12}^\infty \Delta_{12}^0 (\Delta_{12}^{-1/4})^2 \Delta_{12}^{-1/2} \Delta_{12}^{-1/6}, & \Delta_{6,4,12-} &:= \Delta_{12}^\infty \Delta_{12}^0 (\Delta_{12}^{-1/3})^2 \Delta_{12}^{-1/2} \Delta_{12}^{-1/6}, \\ \Delta_{6,5,12-} &:= \Delta_{12}^\infty \Delta_{12}^0 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} (\Delta_{12}^{-1/6})^2, & \Delta_{6,6,12-} &:= \Delta_{12}^\infty \Delta_{12}^0 \Delta_{12}^{-1/3} \Delta_{12}^{-1/4} (\Delta_{12}^{-1/2})^2. \end{aligned}$$

Now, we have

$$\begin{aligned} M_k(\Gamma_0(12)) &= \mathbb{C} E_{k,12}^\infty \oplus \mathbb{C} E_{k,12}^0 \oplus \mathbb{C} E_{k,12}^{-1/3} \oplus \mathbb{C} E_{k,12}^{-1/4} \oplus \mathbb{C} E_{k,12}^{-1/2} \oplus \mathbb{C} E_{k,12}^{-1/6} \oplus S_k(\Gamma_0(12)), \\ S_k(\Gamma_0(12)) &= (\mathbb{C} \Delta_{6,1,12-} \oplus \mathbb{C} \Delta_{6,2,12-} \oplus \cdots \oplus \mathbb{C} \Delta_{6,6,12-} \oplus \mathbb{C} (\Delta_{12})^2) M_{k-6}(\Gamma_0(12)), \end{aligned}$$

for every even integer $k \geq 4$. Then, we have

$$\begin{aligned} M_{6n}(\Gamma_0(12)) &= \mathbb{C} (E_{6,12}^\infty)^n \oplus \mathbb{C} (E_{6,12}^\infty)^{n-1} \Delta_{6,1,12-} \oplus \mathbb{C} (E_{6,12}^\infty)^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,1,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (E_{6,12}^0)^n \oplus \mathbb{C} (E_{6,12}^0)^{n-1} \Delta_{6,2,12-} \oplus \mathbb{C} (E_{6,12}^0)^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,2,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (E_{6,12}^{-1/3})^n \oplus \mathbb{C} (E_{6,12}^{-1/3})^{n-1} \Delta_{6,3,12-} \oplus \mathbb{C} (E_{6,12}^{-1/3})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,3,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (E_{6,12}^{-1/4})^n \oplus \mathbb{C} (E_{6,12}^{-1/4})^{n-1} \Delta_{6,4,12-} \oplus \mathbb{C} (E_{6,12}^{-1/4})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,4,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (E_{6,12}^{-1/2})^n \oplus \mathbb{C} (E_{6,12}^{-1/2})^{n-1} \Delta_{6,5,12-} \oplus \mathbb{C} (E_{6,12}^{-1/2})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,5,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (E_{6,12}^{-1/6})^n \oplus \mathbb{C} (E_{6,12}^{-1/6})^{n-1} \Delta_{6,6,12-} \oplus \mathbb{C} (E_{6,12}^{-1/6})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C} \Delta_{6,6,12-} (\Delta_{12})^{2(n-1)} \\ &\quad \oplus \mathbb{C} (\Delta_{12})^{2n}, \end{aligned}$$

$$\begin{aligned} M_{6n+2}(\Gamma_0(12)) &= E_{2,12+}' M_{6n}(\Gamma_0(12)) \\ &\quad \oplus (\mathbb{C} (E_{1,12+12}')^2 \oplus \mathbb{C} (E_{1,12+3}')^2 \oplus \mathbb{C} (\Delta_{12}^\infty)^4 \oplus \mathbb{C} (\Delta_{12}^0)^4) (\Delta_{12})^{2n}, \end{aligned}$$

$$\begin{aligned}
M_{6n+4}(\Gamma_0(12)) = & E_{4,12}^\infty (\mathbb{C}(E_{6,12}^\infty)^n \oplus \mathbb{C}(E_{6,12}^\infty)^{n-1} \Delta_{6,1,12-} \oplus \mathbb{C}(E_{6,12}^\infty)^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus E_{4,12}^0 (\mathbb{C}(E_{6,12}^0)^n \oplus \mathbb{C}(E_{6,12}^0)^{n-1} \Delta_{6,2,12-} \oplus \mathbb{C}(E_{6,12}^0)^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus E_{4,12}^{-1/3} (\mathbb{C}(E_{6,12}^{-1/3})^n \oplus \mathbb{C}(E_{6,12}^{-1/3})^{n-1} \Delta_{6,3,12-} \oplus \mathbb{C}(E_{6,12}^{-1/3})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus E_{4,12}^{-1/4} (\mathbb{C}(E_{6,12}^{-1/4})^n \oplus \mathbb{C}(E_{6,12}^{-1/4})^{n-1} \Delta_{6,4,12-} \oplus \mathbb{C}(E_{6,12}^{-1/4})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus E_{4,12}^{-1/2} (\mathbb{C}(E_{6,12}^{-1/2})^n \oplus \mathbb{C}(E_{6,12}^{-1/2})^{n-1} \Delta_{6,5,12-} \oplus \mathbb{C}(E_{6,12}^{-1/2})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus E_{4,12}^{-1/6} (\mathbb{C}(E_{6,12}^{-1/6})^n \oplus \mathbb{C}(E_{6,12}^{-1/6})^{n-1} \Delta_{6,6,12-} \oplus \mathbb{C}(E_{6,12}^{-1/6})^{n-1} (\Delta_{12})^2 \oplus \cdots \oplus \mathbb{C}(\Delta_{12})^{2n}) \\
& \oplus (\mathbb{C}(\Delta_{12}^\infty)^4 (\Delta_{12}^0)^4 \oplus \mathbb{C}(\Delta_{12}^{-1/3})^4 (\Delta_{12}^{-1/4})^4) (\Delta_{12})^{2n} \oplus \mathbb{C}E_{1,12+12}' (\Delta_{12})^{2n+1}.
\end{aligned}$$

Furthermore, we can write

$$M_{2n}(\Gamma_0(12)) = \mathbb{C}(\Delta_{12}^\infty)^{4n} \oplus \mathbb{C}(\Delta_{12}^\infty)^{4n-1} \Delta_{12}^0 \oplus \cdots \oplus \mathbb{C}(\Delta_{12}^0)^{4n}.$$

Hauptmodul. We define the *hauptmodul* of $\Gamma_0(12)$:

$$\begin{aligned}
(286) \quad J_{12} &:= \Delta_{12}^0 / \Delta_{12}^\infty (= \eta^3(z) \eta^{-2}(2z) \eta^{-1}(3z) \eta(4z) \eta^2(6z) \eta^{-3}(12z)) \\
&= \frac{1}{q} - 3 + 2q + q^3 - 2q^7 - \cdots,
\end{aligned}$$

where $v_\infty(J_{12}) = -1$ and $v_0(J_{12}) = 1$. Then, we have

$$(287) \quad J_{12} : \partial \mathbb{F}_{12} \setminus \{z \in \mathbb{H} ; \operatorname{Re}(z) = \pm 1/2\} \rightarrow [-6, 0] \subset \mathbb{R}.$$

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